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Mathesis

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Synopsis Palmariorum Matheseos :
OR, A
New Introduction
TO THE
MATHEMATICS:
Containing the
PRINCIPLES
OF
Arithmetic & Geometry

DEMONSTRATED,
In a Short and Easie Method ;

WITH

Their Application to the most Useful Parts thereof: As, Resolving of *Equations*, *Infinite Series*, Making the *Logarithms*; *Interest*, *Simple* and *Compound*; The Chief Properties of the *Conic Sections*; *Mensuration* of *Surfaces* and *Solids*; The Fundamental Precepts of *Perspective*; *Trigonometry*; The *Laws* of *Motion* apply'd to *Mechanic Powers*, *Gunnery*, &c.

Design'd for the Benefit, and adapted to the Capacities
of BEGINNERS.

By W. JONES.

LONDON: Printed by J. Matthews for Jeff. Wale
at the Angel in St. Paul's Church-Yard, 1706.

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Staatsbibliothek
München**

TO THE
HONOURABLE
William Lowndes, Esq;
SECRETARY
TO THE
TREASURY.

SIR,

IN this Attempt, I fear, I have
incurr'd a double Censure :
One by Inscribing a Small Tract
A 2 of

DEDICATION.

of ARITHMETIC, to so Great a Master in Numbers; And another by throwing in my poor endeavors among the Public Affairs, and Crowd of Transactions of the Highest Concern, which Continually pass thro' Your Hands.

I shou'd, therefore, offer at an Apology for my presumption; And entreat Your favorable Interpretation of my performances.

I might inform You too, Sir, of the Forwardness of my Zeal to produce something more worthy Your Acceptance, by a future Progress in these Studies.

My Devotion would lead me farther: But remembering how Valuable every Moment must be to a Person in Your Station, and to One that so Faithfully, so Exactly discharges the Important Business of it; I only crave the
Honor

DEDICATION.

Honor to Declare, how Ambitious I am of shewing my just Esteem and Respect, and how Proud to subscribe my self,

S I R,

Your most obedient,

most humble Servant,

W. Jones.

THE PREFACE.

THE greatest part of the following Sheets was drawn up, and put into this Form, for the Use of some Friends, who had neither Leisure, Conveniency, nor, perhaps, Patience to search into so many different Authors, and turn over so many tedious Volumes, as is unavoidably requir'd to make but a tolerable Progress in the Mathematics: And the Benefit receiv'd by the Method we have taken, encourag'd us to Publish the same, for the Use of Learners, whether inclin'd to the Study of them, or occasionally oblig'd to be Employ'd in the Practice of some part of this Science; whose Principles, therefore, they are concern'd to be well acquainted with, but are unwilling to be incumber'd with more than is just necessary. The Particulars insisted upon in this Treatise are as follow.

1. The Fundamental Rules of Numeral Computation or Common Arithmetic in whole Numbers are briefly, but fully deliver'd; with the necessary Examples in the several Cases; not omitting the most Simple Compendiums in the Operations, that might be of any Use to Practitioners.

2. The First Principles of Literal Computation, usually called Algebra, are illustrated with Variety of Instances, and the Reasons of the several Operations.

3. The usual Methods, as well of Raising Simple Powers, as of Extracting the Roots of such Powers, with Examples both in Numbers and Species.

4. The Nature and Properties of Proportion; where we have bin designedly large, adapting, as far as possible, every thing to the Capacities of young Beginners, that they might not be so seek for their Chief Materials, in prosecuting these Studies: For the whole Body of Mathematics is, in reality, nothing else, but the Doctrine of Proportion; since it only comprehends whatever admits of Greater or Less, as such.

5. Fra-

THE PREFACE.

5. *Fractions, Vulgar and Decimal; (being only Operations about Proportions) where we have bin no less particular, well knowing, that more than an Obscure Conception of these Things is required in him that deserves the Name of an Accomptant. And that the Reader might be compleatly furnish'd with what is necessary to the Understanding the following part of this Treatise, we have given Variety of Examples both Numeral and Literal in the several Cases*

6. *The Arithmetic of Surd or Incommensurable Roots is here made plain and easie: Being Operations frequently used in the Practice of Algebra.*

7. *Some general Directions are laid down for the Solution of Mathematical Problems; With the Methods of Reducing Equations, Exemplified in all the Cases; As also the Derivation, Composition, and Solution of Equations. And the Method used, as well in the few easie Instances given in Simple Equations, as in the Ordinary Rules of Common Arithmetick (which fall under that Head) if well understood and practis'd, is sufficient to enable the Reader for the Solution of any Question whatever relating to any of these Rules, and that after the most Compendious Manner.*

8. *Arithmetical Progression, with the Solution of the several Cases, and the Operations as Length; that the Learner might be farther acquainted with Analytical Investigations, and the Method of proceeding in the like Calculations. Together with some not unuseful Rules relating to the Arithmetic of Infinites; by which we have, in its proper place, demonstrated, briefly and directly, some of the most considerable Propositions in Geometry. Whence also we have explain'd the Nature and Properties of Figurate Numbers; and shewn the Method of Summing up a Series of such Numbers. From these of course are deriv'd the General Theorems for Extracting the Root of any Binomial or Infinitinomial Power, or for Raising such a Power; On which we insisted the more, because the Method of Approximations by Infinite Series depends thereon: Besides, many valuable Rules and Contrivances for the more ready Solution of some of the greatest Difficulties in Mathematics are bus as so many immediate Consequences drawn from hence; and, indeed most of what we call Late Improvements, or New Methods of Investigations, are little else, than the due Application of these Rules.*

The PREFACE.

A Remarkable Instance of the Use of this Theorem is given, in the next place, for making those Numbers called Logarithms, with several Examples, and such Explications of the Nature of them, as may render their Construction and Use intelligible to ordinary Capacities. Other Eminent Theorems drawn from hence, are those for Extracting the Roots of Finite and Infinite Equations; The former we endeavour'd to explain from the successive curious Improvements of Mr. Raphson, Mr. Halley, and Mr. Sharp, with such Illustrations and easy Directions, as might compleat the Learner in the Management of Natural Numbers, by the most expeditious and ready Methods.

9. Increasing and Decreasing Geometric Progressions, with the Solution of the several Cases therein, and the Operations at Length.

10. Interest, Simple and Compound, solv'd in all its Cases.

11. Combinations, Elections, Permutations, and Compositions of Quantities fully handled.

12. The Rudiments of Geometry, wherein the most valuable and useful Propositions in Euclid, Archimedes, Apollonius, and others, are demonstrated in a Natural, Concise, and Easy Method; The principal Properties of (those Figures called) the Conic-Sections are consider'd both with, and without respect to the Cone. Together with Rules for the Mensuration of Lengths, Surfaces, and Solids, investigated by the common general Methods; with the Application of some of those Rules to Cask-Gauging: In the Instances there given, some Quantities (as the Learned Dr. Wallis look'd upon 'em) are determin'd by Summing up their Elements, by the Arithmetic of Infinities; Some (as the Incomparable Sir I. Newton consider'd 'em) by the Velocities of the Motion or Increments, by which they are Generated; according as they render'd the matter either more short, easy, or general: And, for a further Illustration, some by both these Methods.

13. The Principles of Projection, containing the Fundamental Rules for the Practice of Perspective, or for representing any Object as it appears to the Eye in any given Situation: with the Laws of the Orthographic, Stereographic, and Gnomonic Projection of the Sphere, by which any one may readily Project the same upon the Plane of any great Circle, or the several Cases of Spheric Triangles, and measure any Arc or Angle when Projected; As also Delineate any Dial, and with Ease describe the Parallels of the Sun's

The P R E F A C E.

Sun's Declination, or any other Furniture, on the Plane thereof.

14. *Trigonometry both Plane and Spherical, with the Rules necessary for the Solution of any Case therein, briefly demonstrated.*

15. *The Principles of Mechanics, with the general Laws of Motion and their Application in explaining the Powers of Simple Machines and Engines ; With some Select Theorems from Galileus, Sir Is. Newton, and Hugenius, relating to the Motion of Pendulums, Centripetal Forces, Centres of Gravity, &c. To which is added the Doctrine of the Motion of Projects, particularly applied to Gunnery and Throwing of Bombs ; with Directions how to lay a Gun or Mortar to pass so as to strike a Mark with the greatest Certainty and Advantage : where we endeavor'd to follow the Steps of the Learned Geometer Mr. Halley. As also the Common Principles of Optics, wherein the chief Properties of Refracted and Reflected Rays are briefly deliver'd ; with that useful and general Rule, given by the last mention'd Excellent Person, for the Principal Foci of Refracted, or Reflected Rays, on any Figur'd Surface.*

The whole is perform'd with as much Perspicuity and Plainness, as the Subject and our Limits wou'd admit ; And tho' we have not bin over nice in ranging the Particulars of this Treatise, yet we carefully observ'd the Method used by the most Eminent Mathematicians, who in their Writings were pleas'd to condescend to the Capacities of Beginners, as, more especially, in the Arithmetical part, Vieta, Oughtred, Tacquet, and Wallis. Nor have we bin less wanting in consulting the Works of the most Celebrated Ancient and Modern Geometers, that thereby no considerable and useful Proposition or Observation already publish'd, might escape our Notice. We have every where endeavor'd to express things after the clearest and most intelligible manner ; and at the same time have avoided all unnecessary Curiosities that might Cloy the Fancy, or Burden the Memory. We likewise shun'd the tedious Pomp of Linear Demonstrations, affected by some, in things purely Arithmetical, (as
those,

P R E F A C E.

those, which depend upon Proportion and the Common Affections of Quantities in general, are reckon'd to be ;) since Analytical Demonstrations are not only more General and Abstract, and therefore more Universally applicable to Particular Occasions, but also more Plain and Simple and altogether as Scientific, as those made by Lines and Figures: Nor is there any Difficulty from such to disguise the matter so as to make it look more Geometrical. And since the Excellency of that Science, which requires Attention, is its being short and plain ; we therefore have in every thing studi'd Brevity with Perspicuity : So that we doubt not, but a Learner, that does not want the necessary Qualifications of Diligence and Industry, will find, with Advantage, That the whole is (as design'd) A Compendium of the most Select and Primary Principles of Mathematics, and may serve, at least, as an useful Introduction to further Enquiries.

Thus having given our Impartial Reader a Brief Account of this Treatise, we hope his Candor will prompt him favorably to excuse what is amiss, and amend those Errors which unavoidably attend Things of this Nature.

'Twould be here needless to expatiate on the Usefulness of Mathematics, a Part of Humane Literature, to which all the Concerns of Humane Life are deeply engag'd ; Our Pledsure, Security, and Commerce are almost entirely procur'd, maintain'd and improv'd, by the Means of Civil, Military, and Naval Architecture ; and in these there are Variety of illustrious Instances and surprizingly magnificent Pieces of Art, wherein the Effects of Geometry appear in an extraordinary manner. And since we are destitute of Senses acute enough to discover the exact Bulk, Motion, and Figure of Bodies, on which their Properties depend, being conceal'd from us, either by their Remoteness, Opacity, or Transparency : Since therefore, this bounds our ambitious Desires of viewing the more secret Works of Nature, and our Endeavors to subject 'em to the Doctrine of Magnitude and Numbers, whereby we might, beyond the possibility of Opposition or Doubt, Account for the several Phenomena concern'd in our Enquiries ; Yet,

The PREFACE.

Eft quodam prodire tenus, fi non datur ultra ;

and it's somewhat of Content to the Inquisitive Mind (who is ever the least satisfi'd with that, whose Cause is most conceal'd) that by Mechanics and Optics, the Natural Abilities are so far Assisted and Improv'd, as to discover Immensely distant, or Extremely small Objects; Hence we are put in a Capacity of making more Correct Estimations, and of forming juster Notions of the Magnitude, Revolutions and Distances of those Suspendious Fabrics of Nature, which are the constant Subject of our Observations.

ER-

ERRATA.

PAge 18. l. 18. r. 24. p. 26, l. 21. r. 648. p. 39. l. 6. *ee*.
p. 43. l. 12. $x^2 - 16$. p. 44. l. 14. *Ec*. p. 47. l. 24.
 $2ac^2$. p. 53. l. 7. ($2 =$. p. 54. l. 9. $3ac^2$. p. 60. l. 8.
 $a + 6d$. *ibid.* l. 27. *dele which*. p. 64. l. 6. 3^d . p. 68. l. 8.
 $a \frac{1}{r}$. p. 86. l. 20. r. 384. p. 94. l. 12. *And* $\frac{1}{4}$. p. 95. l. 12.
r. $\frac{1}{4}$. *ib.* l. 15. *the Denominators*. p. 96. l. 13. x^3
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p. 185. l. 5. $= L, 22$. p. 206. l. 6. *Solution*. p. 211. l. 19.
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l. 12. $: y^2$. *ibid.* l. 19. $r^2 : : 1$. p. 276. l. 30. *Declina-*
tion.

*The Reader is desir'd to correct these ; and
to excuse any other that possibly may
have escap'd Notice.*

Synopsis

SYNOPSIS

Palmariorum Matheſeos.

General Definitions.

I.

THE most part of the Objects of our Knowledge may be considered as Capable of Augmentation and Diminution; and our Idea of Things as far as they have that Capacity, is what we call Quantity: By which Word may be comprehended whatever can be properly said to have Parts.

SCHOLIUM I.

Under this Definition of Quantity, we may rank Extension, Number, Weight, Motion, Time, &c. The one being taken as Greater or Less, Heavier or Lighter, Swifter or Slower, &c. in relation to another of the same kind.

For Great and Small, &c. are only comparative Terms of Things that are Homogeneous.

B

SCHO.

S C H O L I U M 2.

— And since the Primary and most considerable Property of Quantity is a being Capable of *More* or *Less*; Therefore Quantities may be *Added* to, *Subtracted* from, *Multiplied* by one another, and *Divided* into the Parts they contain.

II.

The mutual Relation of two Things of the same kind Compared together, in respect of Quantity, is call'd Ratio; And the Similitude of Ratio's is call'd Proportion.

III.

The Knowledge of these Comparisons of Quantities, or the Relations they have one to another, is what is generally call'd Mathematicks.

IV.

All Quantities have their Parts either Continuous or Discrete, that is, either United or Separated.

And that Quantity, which has its Parts Separated, is call'd Multitude, and is the Subject of Arithmetick; But that Quantity, which has its Parts united, is called Magnitude, and is the Subject of Geometry.

S C H O.

SCHOLIUM 1.

The word *Part* only denotes our manner of conceiving a *Thing* when consider'd in Relation to its *Whole*, and may be taken as an Indivisible (Component) of it; or as that whose further Divisibility is not then enquired into. Hence,

SCHOLIUM 2.

Unit is properly a Name given to any *Quantity*, consider'd as an *Indivisible*; and,

Number is a Collection of *Units*; when those *Units* are look'd upon as *Whole* the *Number* goes under the Name of an *Integer*; but if they are taken as *Parts* of a *Whole*, then it is called a *Fraction*.

The Explication of the Signs and Characters used in this Treatise.

$\left. \begin{array}{c} \\ \vee \\ + \\ \\ \times \\ \vee \\ :: \end{array} \right\}$	Signifies	Equality, or equal to.
		Majority, or greater than.
		Minority, or less than.
		More, or to be added.
		Less, or to be subtracted.
		The Difference or Excess.
		Multiplied by.
$\left. \begin{array}{c} \vee \\ :: \end{array} \right\}$		Radicality.
		Continual Proportion.

Other Signs or Abbreviations of Words that occur, are explain'd in their own Places.

SYNOPSIS
Palmariorum Matheſeos.
 PART I.
 Containing the
 PRINCIPLES
 OF
 NUMERAL and LITERAL
 ARITHMETICK.

SECTION I.
 Of Numeral Integers.

CHAP. I.
 Notation of Numeral Integers.

DEFINITION I.

NUMERAL Notation teaches how to express in $\left\{ \begin{array}{l} \text{Characters} \\ \text{Words} \end{array} \right\}$ any Number
 propos'd in $\left\{ \begin{array}{l} \text{Words} \\ \text{Characters} \end{array} \right\}$.

S C H O-

S C H O L I U M.

1. The Characters used to express Numbers by, are either

The Ten Numeral Figures of the Arabians :		Or	The Seven Numeral Letters of the Romans.	
Marks	Names		Marks	Names
1	One		I	One
2	Two		V	Five
3	Three		X	Ten
4	Four		L	Fifty
5	Five		C	Hundred
6	Six		D	Five Hundred
7	Seven		M	Thousand
8	Eight			
9	Nine			
0	Nothing or Cypher.			

2. Each of which Figures, besides their own single Value, receives several Denominations, according to their Place and Order.

3. And a Number has so many Places, as there are Figures in it; as, 36487 is a Number of five Places.

4. The Order in whole Numbers, is from the Right to the Left.

The Value of Places increases in a Decuple Proportion; for every Place to the Left, is Ten times the Value of the next Place to the Right.

6. Each Place also has its Name; and those Names, for the more easie reading of large Numbers, are distinguish'd by Periods, half Periods, &c.

For as a Place } is { Ten
So a half Period } is { Thousand } times the Value of that
And a Period } is { Million } before it.

7. A Cypher is of its self insignificant; but by its Place alters the Value of the Subsequent Figure.

8. And

Chap. 1. *Palmariorum Matheſeos.* 7

8. And ſince the *Value* of each *Place* is Ten times the *Value* of the next before it, 'tis certain,

That $\left. \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\}$ in the first Place is $\left. \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\}$ in the 2d. $\left. \begin{matrix} 10 \\ 20 \\ 30 \end{matrix} \right\}$ in the 3d. $\left. \begin{matrix} 100 \\ 200 \\ 300 \end{matrix} \right\}$ &c.

The Order and Names of Periods, &c.				
Periods	Half Periods	Degr.	Places	Order
Units	un	u	1	1
	x	x	2	
	c	c	3	
	th	th	4	
	x	x	5	
	c	c	6	
Millions	un	u	7	2
	x	x	8	
	c	c	9	
	th	th	10	
	x	x	11	
	c	c	12	
Billions	un	u	13	3
	x	x	14	
	c	c	15	
	th	th	16	
	x	x	17	
	c	c	18	
Trillions	un	u	19	4
	x	x	20	
	c	c	21	
	th	th	22	
	x	x	23	
	c	c	24	
&c.				

C O R O L.

C H A P. II.

Addition of Integers.

DEFINITION.

ADDITION is the Collection of ſeveral Numbers or Quantities into one Sum.

PROBLEM 1.

To add Integers of like Name into one Sum.

R U L E.

Place all Numbers of like kind under one another, Add up the Units, and if their Sum be under Ten, ſet that Sum underneath;

if it is equal to Ten, or Tens, ſet a Cypher underneath: if it is above Ten, or Tens, ſet the Exceſs underneath:

And for every Ten, carry an Unit to the next Place, and ſo proceed.

E X A M P L E 1.

$$\begin{array}{r}
 \text{To } 235 \\
 \text{Add } 432 \\
 \hline
 \text{Sum } 667
 \end{array}
 \quad
 \begin{array}{l}
 \left. \begin{array}{l} 200 + 400 \\ 30 + 30 \\ 5 + 2 \end{array} \right\} \text{ i. e. } \left\{ \begin{array}{l} 200 + 400 \\ 30 + 30 \\ 5 + 2 \end{array} \right. \\
 \hline
 = 600 + 60 + 7
 \end{array}$$

The manner of Operation.

$$\begin{array}{l}
 2 \text{ and } 5 = 7 \\
 3 \text{ and } 3 = 6 \\
 4 \text{ and } 2 = 6
 \end{array}
 \left\{ \begin{array}{l} \text{Units} \\ \text{Tens} \\ \text{Hundreds} \end{array} \right.
 \text{ which ſet under }$$

$$\begin{array}{r} 235 \\ 432 \\ \hline \end{array}$$

$$\begin{array}{r} 235 \\ 432 \\ \hline \end{array}$$

For there are $\left\{ \begin{array}{l} 600 \\ 60 \\ 7 \end{array} \right\}$ or thus $\left\{ \begin{array}{l} 7 \\ 60 \\ 600 \end{array} \right\}$ in the given Numbers.

Therefore $667 = 667$ must be their Sum.

E X A M P L E. 2.

To 6985
 Add 7645
 And 4310

 Sum 18940

The manner of Operation.

$\left. \begin{array}{l} 0 \text{ and } 5 \text{ and } 5 = 10 \\ 1 \text{ and } 1 \text{ and } 4 \text{ and } 8 = 14 \\ 1 \text{ and } 3 \text{ and } 6 \text{ and } 9 = 19 \\ 1 \text{ and } 4 \text{ and } 7 \text{ and } 6 = 18 \end{array} \right\} \text{ set down } \left\{ \begin{array}{l} 0 \\ 4 \\ 9 \\ 18 \end{array} \right\} \text{ and carry } 1 \text{ to the next place.}$

6985
 7645
 4310

For there are $\left\{ \begin{array}{l} 10 \text{ Units} \\ 13 \text{ Tens} \\ 18 \text{ Hundreds} \\ 17 \text{ Thousands} \end{array} \right\}$ in the Numbers given.

Therefore 18940 must be the Number required, equal to all those given.

Since those things which are equal amongst themselves, are also equal to one another.

E X-

EXAMPLE III.

To	38796	38796	38796	38796
Add	4638	4638	4638	4638
	30000	14	13	43434
	12000	120	1212	
	1300	1390	3 14	
	120	12000		
	14	30000	43434	
	43434	43434		

PROBLEM II.

To Add Integers of different Names.

RULE.

Place them severally in the same Line, with the Sign $+$ between them, or understood to be so.

EXAMPLES.

1	2	3
To 40	8 b	16 s.
Add 6	35 m.	9 d.
Sum = 40 + 6	8 b + 35 m.	16 s. + 9 d.
i. e. 46.	8 b. 35 m.	16 s. 9 d.

C 2

PRO-

PROBLEM III.

To Add Integers and Parts.

EXAMPLE I.

$$\begin{array}{r}
 \text{To } 5\text{F. } 16\text{I. } \} \text{ or } \{ 6\text{F. } 4\text{I.} \\
 \text{Add } 6. \quad 14. \} \{ 7. \quad 2. \\
 \hline
 \text{Sum } 11. \quad 30. = 13. \quad 6.
 \end{array}$$

EXAMPLE II.

$$\begin{array}{r}
 \text{To } 164\text{l. } 43\text{s. } 6\text{d. } \} \text{ or } \{ 164\text{l. } 15\text{s. } 6\text{d.} \\
 \text{Add } 79. \quad 12. \quad 9. \} \{ 79. \quad 12. \quad 9. \\
 \hline
 \text{Sum } 244. \quad 8. \quad 3. = 243. \quad 27. \quad 15.
 \end{array}$$

CHAP. III.

Subtraction of Integers.

DEFINITION.

SUBTRACTION is the taking of one Number or Quantity from another, to find their Difference.

SCHOLIUM.

The $\left\{ \begin{array}{l} \text{greatest} \\ \text{least} \end{array} \right\}$ of the given Numbers $\left\{ \begin{array}{l} \text{Minuend} \\ \text{Subducent} \end{array} \right\}$. And the Number found is call'd the Remainder or Difference.

P R O B.

PROBL. I.

To Subtract Integers of like Names, when the Superiour Numbers are greater than, or equal to their Inferiours,

RULE.

1. Place the Subducent under the Minuend, and draw a Line under both.
2. Begin at the Right-hand, take the leſs from the greater, or Equals from Equals, and ſet the Difference of each Row underneath.

Example, In Integers alone.

Minuend 638

Subducent 213

Remainder 425

The manner of Operation.

$\left. \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix} \right\}$ from $\left\{ \begin{smallmatrix} 8 \\ 3 \\ 6 \end{smallmatrix} \right\}$ and there remain $\left\{ \begin{smallmatrix} 5 \\ 2 \\ 4 \end{smallmatrix} \right\}$ which ſet below,

That is,

$\left. \begin{smallmatrix} 3 \\ 10 \\ 200 \end{smallmatrix} \right\}$ from $\left\{ \begin{smallmatrix} 8 \\ 30 \\ 600 \end{smallmatrix} \right\}$ remainder is $\left\{ \begin{smallmatrix} 5 \\ 20 \\ 400 \end{smallmatrix} \right\}$

Theref, 213

For ſince the Whole is equal to the Sum of all its Parts, therefore the Subduction of all the Parts is the ſame with the Subduction of the whole.

Examples,

Examples, In Integers and Parts.

Minuend	27 d.	48'	36''.	146 l.	18 s.	6 d.
Subducent	12.	31.	24.	22.	8.	2.
Remainder	15.	17.	12.	124.	10.	4.

P R O B. II.

To Subduct Integers of the same Name, when some of the Superiour Numbers are less than their Inferiour.

R U L E.

1. Place your Numbers, and begin as before.
2. And according to their respective Value, take one of the next Denomination, out of which Subduct; and to the Remainder, add the Superiour, setting their Sum underneath.
3. Then add what you took to the next place on the Left-hand; and so proceed by this, or the former Rule.

Example, In Integers alone.

From 2537
Subd. 1648

Rem. 889

The manner of Operation.

1 and 4 = 5 } from { 17 } Rem. { 9 } which set below
1 and 6 = 7 } { 13 } { 8 }
 } { 15 } { 8 }

That

That is,

$$\begin{array}{r}
 \begin{array}{r} 8 \\ 16 + 40 \\ 100 + 600 \\ 1000 + 1000 \end{array} \left\{ \begin{array}{l} \text{from} \\ 2537 \end{array} \right. \begin{array}{r} 7 + 10 \\ 30 + 100 \\ 500 + 1000 \\ 2000 + 0000 \end{array} \left\{ \begin{array}{l} \text{Rem.} \\ 889 \end{array} \right. \begin{array}{r} 9 \\ 80 \\ 800 \\ 0000 \end{array} \\
 \text{Theref. } 1648
 \end{array}$$

For by saying 8 from 17, I add Ten to the *Minuend*, but I add also the same to the *Subducend*, by saying 1 and 4 = 5; therefore the Remainder must be the same.

The Operation also may be thus;

$$\begin{array}{r}
 \begin{array}{r} 8 \\ 4 \\ 6 \\ 1 \end{array} \left\{ \begin{array}{l} \text{from} \\ 17 \\ 13-1 \\ 15-1 \\ 2-1 \end{array} \right. \left\{ \begin{array}{l} \text{Rem.} \\ 9 \text{ Units} \\ 8 \text{ Tens} \\ 8 \text{ Hundreds} \\ 0 \end{array} \right.
 \end{array}$$

That is,

$$\begin{array}{r}
 \begin{array}{r} 8 \\ 40 \\ 600 \\ 1000 \end{array} \left\{ \begin{array}{l} \text{from} \\ 2537 \end{array} \right. \begin{array}{r} 7 + 10 \\ 30 + 100 - 10 \\ 500 + 1000 - 100 \\ 2000 + 0000 - 1000 \end{array} \left\{ \begin{array}{l} \text{Rem.} \\ 889 \end{array} \right. \begin{array}{r} 9 \\ 80 \\ 800 \\ 0000 \end{array} \\
 \text{Theref. } 1648
 \end{array}$$

For by adding a Ten to the Units, and taking it away from the Tens, the value of the Number is not changed.

Examples, *In Integers and Parts.*

$$\begin{array}{r}
 \text{From } 5 \text{ s. } 3 \text{ d. } \left\{ \begin{array}{l} i. \\ 2. \end{array} \right. \left\{ \begin{array}{l} 5 \text{ s. } 15 \text{ d. } \\ 3. \end{array} \right. \left\{ \begin{array}{l} \text{or} \\ 2. \end{array} \right. \left\{ \begin{array}{l} 4 \text{ s. } 15 \text{ d. } \\ 2. \end{array} \right. \\
 \text{Subd. } 2. \quad 9. \quad \left\{ \begin{array}{l} i. \\ 2. \end{array} \right. \left\{ \begin{array}{l} 3. \end{array} \right. \left\{ \begin{array}{l} \text{or} \\ 2. \end{array} \right. \left\{ \begin{array}{l} 2. \end{array} \right. \\
 \text{Rem. } 2. \quad 6. \quad = \quad 2. \quad 6. \quad = \quad 2. \quad 6.
 \end{array}$$

From

$$\begin{array}{r}
 \text{From. } 246\text{ l. } 3\text{ s. } 4\text{ d. } \} 1. \text{ e. } \{ 246\text{ l. } 23\text{ s. } 16\text{ d. } \} \\
 \text{Subd. } 68. 10. 6. \} \\
 \hline
 \text{Rem. } 177. 12. 10. = 177. 12. 10.
 \end{array}$$

$$\begin{array}{r}
 \text{or } \left\{ \begin{array}{l} \text{From } 245\text{ l. } 22\text{ s. } 16\text{ d.} \\ \text{Subd. } 68. 10. 6. \\ \hline \text{Rem. } 177. 12. 10. \end{array} \right.
 \end{array}$$

THEOREM.

In Subtraction, the Subducent together with the Remainder, is equal to the Minuend.

For all the Parts taken together, are equal to the whole.

And if the Subducent be taken from the Minuend, there rests the Remainder.

But if a Part be taken from the Whole, the Remainder will be the other Part.

Therefore the Subducent, together with the Remainder are all the Parts of the Minuend, and consequently equal to it.

COROLLARY.

Hence; *Addition* and *Subduction*, serve Reciprocally to prove each other.

For Addition and Subduction are opposite in all Cases; and what is done by the one, is undone by the other.

Thus

Thus if to 6	And if from 10
be added 4	Subduct. 4
Sum 10	Rem. 6

That is, if $6 + 4 = 10$, then $10 - 4 = 6$.

CHAP. IV.

Multiplication of Integers.

DEFINITION.

MULTIPLICATION is a manifold Addition, or the repeating a given Quantity, as often as required; That is, to take a Quantity so many Times, Part or Parts of a Time, as is represented by another.

SCHOLIUM

The Num- { to be Repeated } is called { Multiplicand
ber { of Repetitions } the { Multiplier.

The Sum of the Number so often Repeated, is called the *Product*. Both *Multiplicand* and *Multiplier* are call'd *Factors*.

COROLLARY.

Hence, As Unit, is to one Factor:
So is the other Factor, to the Product.

D

CASE

LEMMA. I.

The Product of any two Numbers, is equal to the several Products made by multiplying one of those Numbers by the several Parts of the other.

$$\begin{array}{r} \text{Thus } 9 \left. \begin{array}{l} \text{ } \\ \text{ } \end{array} \right\} \text{i. e. } \left\{ \begin{array}{l} 4 + 3 + 2 \\ 3 \end{array} \right. \quad 34 \left. \begin{array}{l} \text{ } \\ \text{ } \end{array} \right\} \text{i. e. } \left\{ \begin{array}{l} 30 + 4 \\ 2 \end{array} \right. \\ \hline 27 = 12 + 9 + 6 \quad 68 = 60 + 8 \end{array}$$

For, since the *Whole*, and all its *Parts* taken together, make but one and the same thing :

Therefore the *Multiplying* the *Whole*, and all its *Parts*, gives the same *Product*.

CASE II.

To Multiply a Compound Number by a Single one,

EXAMPLE.

$$\begin{array}{r} \text{Mult. } 6452 \left. \begin{array}{l} \text{ } \\ \text{ } \end{array} \right\} \text{or } \left\{ \begin{array}{l} 6000 + 400 + 50 + 2 \\ 3 \end{array} \right. \\ \text{by } 3 \left. \begin{array}{l} \text{ } \\ \text{ } \end{array} \right\} \text{or } \left\{ \begin{array}{l} \text{ } \\ 3 \end{array} \right. \\ \hline \text{Product. } 19356 = 18000 + 1200 + 150 + 6 \end{array}$$

$$\begin{array}{r} \text{That is } 6452 \left. \begin{array}{l} \text{ } \\ \text{ } \end{array} \right\} \text{or } \left\{ \begin{array}{l} 6452 \\ 3 \end{array} \right\} \text{or } \left\{ \begin{array}{l} 6452 \\ 3 \end{array} \right. \\ \hline \begin{array}{r} 11156 \\ 82 \\ \hline 19356 \end{array} \quad \begin{array}{r} 6 \\ 150 \\ 1200 \\ 18000 \\ \hline 19356 \end{array} \quad \begin{array}{r} 18000 \\ 1200 \\ 150 \\ 6 \\ \hline 19356 \end{array} \end{array}$$

$$\text{For since 3 times } \left\{ \begin{array}{r} 6000 \\ 400 \\ 50 \\ 2 \end{array} \right\} \text{ is } \left\{ \begin{array}{r} 18000 \\ 1200 \\ 150 \\ 6 \end{array} \right\}$$

$$\text{Therefore 3 times } 6452 \text{ is } 19356$$

L E M M A II.

The Product of any two Quantities, is equal to the several Products made by Multiplying all the Parts of the one, by all the Parts of the other.

$$\text{Thus } \left\{ \begin{array}{r} 12 \\ 7 \end{array} \right\} \text{ or } \left\{ \begin{array}{r} 10 + 2 \\ 3 + 4 \end{array} \right\} \text{ or } \left\{ \begin{array}{r} 6 + 4 + 2 \\ 3 + 4 \end{array} \right\}$$

$$84 = 36 + 6 + 40 + 8 = 18 + 12 + 6 + 24 + 16 + 8$$

C A S E III.

To Multiply one Compound Number by another,

R U L E

1. Place each Number respectively under its kind.
2. Multiply each Figure of the Multiplicand, by each Figure of the Multiplier; and observe to set the first Figure of each respective Product under that Figure of the Multiplier by which it was made.
3. Add the several Products together for the whole Product.

E X A M P L E I.

Mult. 123
by 23

$$\begin{array}{r} 369 \\ 246 \\ \hline 2829 \end{array}$$

$$\left\{ \begin{array}{r} 123 \\ 20 \end{array} \right\} + \left\{ \begin{array}{r} 123 \\ 3 \end{array} \right\} \text{ or } \left\{ \begin{array}{r} 100 + 20 + 3 \\ 20 + 3 \end{array} \right\}$$

$$2460 + 369 = 2000 + 400 + 60 + 300 + 60 + 9$$

For

$$\begin{array}{l} \text{For } \left\{ \begin{array}{l} 3 \times \\ 20 \times \end{array} \right\} \left\{ \begin{array}{l} 100 = 300 \\ 20 = 60 \\ 3 = 9 \\ 100 = 2000 \\ 20 = 400 \\ 3 = 60 \end{array} \right. \end{array}$$

$$\text{Therefore } 23 \times 123 = 2829$$

E X A M P L E 2.

Multiplicand 5326
Multiplier 427

37282

10652

21304

Product 2274202

The manner of Operation.

$$\text{I. 7 times } \left\{ \begin{array}{l} 6, \\ 2, + 4 = 18 \\ 3, + 1 = 22 \\ 5, + 2 = 37 \end{array} \right\} \left\{ \begin{array}{l} \text{subscribe} \\ 2 \\ 8 \\ 2 \\ 37 \end{array} \right\} \left\{ \begin{array}{l} \text{carry} \\ 4 \\ 1 \\ 2 \end{array} \right\}$$

$$\text{II. 2 times } \left\{ \begin{array}{l} 6, \\ 2, + 1 = 3 \\ 3, \\ 5, = 10 \end{array} \right\} \left\{ \begin{array}{l} \text{subscribe} \\ 2 \text{ and carry } 1, \\ 5 \\ 6 \\ 10 \end{array} \right\}$$

$$\text{III. 4 times } \left\{ \begin{array}{l} 6, \\ 2, + 2 = 10 \\ 5, \end{array} \right\} \left\{ \begin{array}{l} \text{subscribe} \\ 4 \\ 0 \end{array} \right\} \left\{ \begin{array}{l} \text{carry} \\ 2 \\ 1 \end{array} \right\}$$

Multi-

		Multiplicand	5326
		Multiplier	427
For since	7 X	6 =	42. 1st.
		20 =	140. 2
		300 =	2100. 3
		5000 =	35000. 4
	20 X	6 =	120. 5
		20 =	400. 6
		300 =	6000. 7
		5000 =	100000. 8
	400 X	6 =	2400. 9
		20 =	8000. 10
		300 =	120000. 11
		5000 =	2000000. 12

Therefore 2274202 is the Sum of the *Products* of all the *Parts* of the *Multiplicand* 5326, Multiplied by all the *Parts* of the *Multiplier* 427, and consequently equal to 5326 X 427, by *Lemma 2*.

SCHOLIUM I.

When either the *Multiplicand*, *Multiplier*, or both, have *Cyphers* towards the *Right-hand*; then *Multiply the Significant Figures by the former Rules, and annex to the Product as many Cyphers as there are in the Multiplicand and Multiplier.*

EXAMPLES.

$$246 \text{ by } \left\{ \begin{array}{l} 10 \\ 100 \\ 1000 \\ 10000 \end{array} \right\} \text{ gives } \left\{ \begin{array}{l} 2460 \\ 24600 \\ 246000 \\ 2460000 \end{array} \right.$$

143	9520	624000
20	3400	4300
2860	3808	1872
	2856	2496
	32368000	2683200000

SCHOLIUM 2.

When there are Cyphers in the Multiplier; then *Sub-*
scribe those Cyphers in Order, before the particular Pro-
duct of the next Multiplier by the Multiplicand.

E X A M P L E.

427	6700042
306	100005
2562	33500210
12810	67000420000
130662	670037700210

SCHOLIUM 3.

To Multiply by any Compound Number under 20;
you need only,

Set the Product made by the Unit Figure of the Multiplier,
a Place further to the Right-hand, and add thereto the
Multiplicand. Thus,

Multipli-

$$\begin{array}{r}
 \text{Multiplicand} \quad 25436 \\
 \text{Multiplier} \quad 14 \\
 \hline
 101744 \text{ 1st. Product} \\
 \hline
 \text{Product} \quad 356104
 \end{array}$$

Or shorter, thus.

$$\begin{array}{r}
 25436 \\
 14 \\
 \hline
 356104
 \end{array}$$

The manner of Operation.

$$\begin{array}{l}
 6 = 24 \\
 3 = 12, + 2 + 6 = 20 \\
 4 = 16, + 2 + 3 = 21 \\
 5 = 20, + 2 + 4 = 26 \\
 2 = 8, + 2 + 5 = 15 \\
 \text{Then} \quad 2 + 1 = 3
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \text{4 by}$$

$$\begin{array}{l}
 4 \\
 0 \\
 1 \\
 6 \\
 5 \\
 3
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \text{Subtr.}$$

$$\begin{array}{l}
 2 \\
 2 \\
 2 \\
 2 \\
 1
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \text{carry}$$

SCHOLIUM 4.

If a Quantity be Multiplied by the Component Parts of the Multiplier, the Product will be the same as if it had been Multiplied by the Multiplier it self.

Thus, 245 by 7, and the Product by 6, is the same as if 245 was Multiplied by 7×6 , that is, by 42.

CHAP.

C H A P. V.

Division of Integers.

DEFINITION.

DIVISION is a *Manifold Subduction*; or the taking of one Number or Quantity out of another, as often as possible.

As 6 Divided by 2 gives 3.

For $6 - 2 = 4 - 2 = 2 - 2 = 0$.

S C H O L I U M.

The Number $\left\{ \begin{array}{l} \text{to be Divided} \\ \text{Dividing} \end{array} \right\}$ is called $\left\{ \begin{array}{l} \text{Dividend} \\ \text{Divisor} \end{array} \right\}$

And the Number of Times they contain each other, is called *Quotient*.

C O R O L L A R Y.

Hence, as the *Divisor*, is to the *Dividend*:

So is *Unit*, to the *Quotient*.

The *Terms* in *Division* are thus Placed.

Divisor *Dividend* *Quotient*

2) 6 (3

Or,

Dividend $\frac{6}{2}$ (*3 Quotient*.
Divisor

E

PROB.

P R O B L E M.

To Divide one Number by another.

R U L E.

1. Set a Point under the last of the Left-hand Places in the Dividend, out of which the Divisor may be taken; and the Number of Places to the Right of that Point inclusive, gives the Number of Places in the Quotient.
2. Try how often you can take the Divisor out of that first part of the Dividend, setting the Number of Times in the Quotient; then Multiply the Divisor thereby, Subtract the Product out of the said part of the Dividend, and subscribe the Remainder.
3. To the Right of the Remainder, set the next Figure of the Dividend, from which take the Divisor as often as you can, setting the Number of Times in the Quotient, Multiply the Divisor thereby, Subtracting the Product as before; and in this manner the Operation must be repeated, till you come to the end.

E X A M P L E I.

Divisor	Dividend	Quotient
2)	642	(321

642
—
64
—
8
—
0
—
0
—

for

For 2 in $\left\{ \begin{array}{l} 600 \\ 40 \\ 8 \end{array} \right\} \left\{ \begin{array}{l} 300 \\ 20 \\ 4 \end{array} \right\}$ times

Therefore 2) 648 (324 times

E X A M P L E. II.

3) 19358 (6452

```

    18...
    ---
    13..
    12..
    ---
    15.
    15.
    ---
    98
    96
    ---

```

Rem. 2

For 3 in $\left\{ \begin{array}{l} 18000 \\ 1200 \\ 150 \\ 8 \end{array} \right\} \left\{ \begin{array}{l} 6000 \\ 490 \\ 50 \\ 2 \end{array} \right\}$ times

Rem. 2

Therefore 3) 19358 (6452

E X A M P L E.

EXAMPLE III.

Divisor Dividend Quotient

24) 5664 (236

...

48

86

72

144

144

000

The manner of Operation.

1. Having placed the Numbers, and pointed them as the Rule Directs: and finding I can have 24 in 56 but 2 times, therefore I set 2 in the *Quotient*, the *Divisor* Multiplied thereby, gives 48, which Subducted from 56 leaves 8; to the Right of 8 I set the next Figure of the *Dividend*, 6.

2. I can have 24 in 86 thrice, therefore I set 3 in the *Quotient*; then thrice 24 is 72, which Subducted from 86 leaves 14; to the Right of 14, I set 4 the next and last Figure of the *Dividend*.

3. I can have 24 in 144, 6 times, therefore I set 6 in the *Quotient*; then 6 times 24 is 144, which Subducted from 144, leaves nothing. Hence, I conclude, That 24 is contained 236 times in 5664.

The

The Reason of this Operation is evident from the following Procefs.

$$\begin{array}{r} \text{I.} \quad \begin{array}{r} 24 \text{ in) } 5664 \\ 200 \times 24 = 4800 \end{array} \end{array}$$

$$\begin{array}{r} 800 \\ + 60 \\ \hline \end{array}$$

$$\begin{array}{r} \text{II.} \quad \begin{array}{r} 24 \text{ in) } 860 \\ 30 \times 24 = 720 \end{array} \end{array}$$

$$\begin{array}{r} 140 \\ + 4 \\ \hline \end{array}$$

$$\begin{array}{r} \text{III.} \quad \begin{array}{r} 24 \text{ in) } 144 \\ 6 \times 24 = 144 \end{array} \end{array}$$

$$\begin{array}{r} \text{And fince} \left\{ \begin{array}{l} 24) 4800 (200 \\ 24) 720 (30 \\ 24) 144 (6 \end{array} \right\} \text{times} \end{array}$$

$$\text{Therefore } 24) 5664 (236$$

S C H O L I U M I.

If the *Divisor* be greater than the *Dividend*; or if they be of different Names, then, *The Dividend set above, and the Divisor below a Line, represent the Quotient.*

Ex. I. 4 Divided by 9, the Quotient is $\frac{4}{9}$

II. 24 Feet Divided by 6 Yards.

The Quotient is $\frac{24 \text{ Feet}}{6 \text{ Yards.}}$

S C H O.

S C H O L I U M. 2.

If the *Divisor* have *Cyphers* towards the Right Hand; then,

Cut off so many of the Right Hand Places of the *Dividend* as there are *Cyphers* in the *Divisor*, which annex to the *Remainder*, when the Operation is finished.

E X A M P L E S.

I.	10)	267 8	(267 $\frac{8}{10}$)
II.	300)	69 52	(23 $\frac{52}{300}$)
III.	3200)	2457 26	(76 $\frac{2457}{3200}$)
		..	
		224	
		217	
		192	
		25	

T H E O R E M.

The *Divisor* Multiplied by the *Quotient* is equal to the *Dividend*.

As suppose 15 was Divided by 3, the *Quotient* will be 5.

But the *Quotient* 5 is so many Parts of the *Dividend* 15, as Unit is of the *Divisor* 3, that is, the third Part; by Defin. of Division.

Therefore, If the *Quotient* be taken so many times as there are Units in the *Divisor* 3, the Sum will be equal to the *Dividend* 15.

And all the Parts (5 times 3) taken together are equal to the whole (15).

Therefore, The *Product* of the *Divisor* by the *Quotient* is equal to the *Dividend*.

C O R O L L.

C O R O L L A R Y.

Hence, 'tis evident, That *Multiplication* and *Divison* serve Reciprocally to prove each other.

For *Multiplication* and *Divison* are two contrary Operations, and what is done by the one, is undone by the other.

Thus, if 5 Multiplied by 3, give 15.

Then 5 is contain'd 3 times in 15.

Therefore $\frac{15}{3} (= 5)$ and $\frac{15}{5} (= 3)$.

Also if 15 Divided by 3 give 5,

Then 5 taken 3 times is 15.

Therefore $5 \times 3 = 15 = 3 \times 5$.

SECT.

S E C T. II.

Of Literal or Algebraic Integers.

C H A P. I.

Notation of Algebraic Integers.

DEFINITION.

ANY Number or Quantity whatsoever, whether known or unknown, may be universally express'd by Notes, Characters or Letters of the Alphabet, at Pleasure : And this way of Notation, applied to Arithmetical Operations, is what is commonly call'd ALGEBRAICAL or LITERAL ARITHMETIC.

S C H O L I U M.

1. In Operations perform'd by *Literal Arithmetic*, all Quantities, known or unknown, are represented by Letters, with prefix'd Marks or Signs, whereby, (according to the Nature of the Proposition) they may be so order'd by Addition, Subduction, Multiplication, Division, &c. as if each particular Part was actually known; so that the

the Position and Relation of one Quantity to another is visible thro' the whole Course of the Process, and consequently that of the *known* to the *unknown*.

2. Quantities represented by $\left\{ \begin{smallmatrix} \text{the same} \\ \text{different} \end{smallmatrix} \right\}$ Letters in any Operation, are supposed to be of $\left\{ \begin{smallmatrix} \text{the same} \\ \text{different} \end{smallmatrix} \right\}$ Value.

3. And between Quantities of *different values*, there may be an *Equality*, as $6a = 2b$; 6 Feet = 2 Yards.

4. The number of Times any Quantity is taken, must be prefix'd to it, and is call'd a *Coefficient*, or *Co-factor*.

5. A Quantity without a $\left\{ \begin{smallmatrix} \text{Coefficient} \\ \text{Sign} \end{smallmatrix} \right\}$ is supposed to have $\left\{ \begin{smallmatrix} \text{Unit} \\ \text{the Sign} \end{smallmatrix} \right\} +$ prefix'd to it.

6. *Simple* Quantities, are $\left\{ \begin{smallmatrix} \text{Simple} \\ \text{Compound} \end{smallmatrix} \right\}$ those which $\left\{ \begin{smallmatrix} \text{have but one Member.} \\ \text{are connected by + and -} \end{smallmatrix} \right\}$.

And since *Addition*, *Subduction*, *Multiplication* and *Division*, are the Common Affections of all Quantities, therefore we shall in the next place endeavour with all the Brevity and Plainness possible, to apply these Rules to *Letters*, as we have to *Numbers*.

A X I O M S.

1. If *to* or *from* equal Quantities, equal ones be *Added* or *Subducted*, their *Sum* or *Remainder* will be equal.

2. If equal Quantities be *Multiplied* or *Divided* by equal ones, their *Products* or *Quotients* will be equal.

C H A P. II.

Addition of Algebraic Integers.

C A S E. I.

When Quantities have the same Name, and the same Sign,

R U L E.

Let the propos'd Quantities be Collected, and their respective Signs adjoy'n'd.

E X A M P L E S.

To	$+a$	$+2b$	$-2x$	$yy + 2x - 2$
Add	$+a$	$+5b$	$-x$	$yy + 3x - 5$
Sum	$+a + a$	$2b + 5b$	$-3x$	$2yy + 5x - 7$
Or	$+2a$	$+7b$		

For 'tis manifest from the Common way of Numbering, that $2 + 5 = 7$, of any thing of like Name; as 2 Miles and 5 Miles are 7 Miles.

C A S E II.

When Quantities have the same Name, but different Signs,

R U L E.

Let them be Subducted from each other, and the Sign of the Greater adjoy'n'd to the Remainder.

E X

E X A M P L E S.

To	$+ 3a$	$- 3x$	$+ 4z$	$6aa - x + 30$
Add	$- 2a$	$+ 2x$	$- 4z$	$x - 20 - 6aa$
Sum	$+ 3a - 2a$	$- 3x + 2x$	$+ 4z - 4z$	$+ 10$
Or	$+ a$	$- x$	0	

For to Add a *Negative*, is to take away a *Positive* ;
Therefore, to Connect a *Negative* and a *Positive*, is to
make the one and destroy the other.

Thus, If *A* has 600*l.* and owes 400*l.* 'tis plain that
the Sum, or his Worth is but 200*l.*

And if *A* has 600*l.* and owes 900*l.* then the Worth
is — 300*l.* or 300*l.* worse than nothing.

C A S E III.

When Quantities are of different Names,

R U L E.

*Let them be set down in order, with their own Signs
prefix'd.*

E X A M P L E S.

To	a	$5b$	$a + b$	$3a - 2z - 3x + 6$
Add	c	$4m$	$c - d$	$47 + pp + 3a + 2z$
Sum	$a + c$	$5b + 4m$	$a + b + c - d$	$6a + 53 - 3x + pp$

For 4 *Miles* and 5 *Hours*, make neither 9 *Miles* nor 9 *Hours*.

C H A P. III.

Subduction of Algebraic Integers.

Case $\left\{ \begin{array}{l} 1. + \text{ from } +; \text{ and } - \text{ from } - \\ 2. - \text{ from } +; \text{ and } + \text{ from } - \end{array} \right\}$

Both perform'd by this

General R U L E.

Let (or suppose) all the Signs of the Subducend be changed, Then the Quantities Collected, (as in Addition) give the Remainder.

E X A M P L E S In Case 1.

$$\begin{array}{r|l|l|l}
 \text{From} & +3a & +ax & -2b \\
 \text{Subd.} & +2a & +ax & -2b \\
 \hline
 \text{Rem.} & +3a-2a & ax-ax & -2b+2b \\
 \hline
 \text{Or} & +a & 0 & 0
 \end{array}
 \quad
 \begin{array}{r|l|l|l}
 5n+4r+6 & & & \\
 60+3r+9 & & & \\
 \hline
 5n+r-63 & & &
 \end{array}$$

For to *take away* any *Thing*, is the same as to *Subjoin* the *Defect* of that *Thing*.

Therefore, to $\left\{ \begin{array}{l} \text{Affirmation} \\ \text{Negation} \end{array} \right\}$ of any thing, $\left\{ \begin{array}{l} \text{Negative} \\ \text{Affirmative} \end{array} \right\}$ take away the $\left\{ \begin{array}{l} \text{Negation} \\ \text{Affirmation} \end{array} \right\}$ is to make it $\left\{ \begin{array}{l} \text{Affirmative} \\ \text{Negation} \end{array} \right\}$.

E X A M P L E S, In Case 2.

$$\begin{array}{r|l|l|l}
 \text{From} & +3a & -3x & +6a \\
 \text{Subd.} & -2a & +2x & -2a-y \\
 \hline
 \text{Rem.} & +5a & -5x & +8a+y
 \end{array}
 \quad
 \begin{array}{r|l|l|l}
 3aa-5a+7 & & & \\
 aa+a-2 & & & \\
 \hline
 2aa-6a+9 & & &
 \end{array}$$

For to *take away* the want of a *Thing*, is to *Add* that very *Thing*, by taking away the *Negation* of it.

But to *take away* the *Being* or *Affirmation* of a *Thing*, must necessarily produce the *Want* or *Negation* of that *Thing*.

S C H O.

S C H O L I U M.

In *Addition* and *Subduction*, it is indifferent as to Order, how the ſeveral Quantities do ſtand, ſo that each has its own Sign.

$$\begin{aligned} \text{For } x - y + z &= x + z - y = -y + z + x = z + x - y = x \\ \text{Or } 8 - 6 + 2 &= 8 + 2 - 6 = -6 + 2 + 8 = 2 + 8 - 6 = 4 \end{aligned}$$

C H A P. IV.

Multiplication of Algebraic Integers.

N O T E.

That in *Mul-riplication* $\left\{ \begin{array}{l} \text{Like} \\ \text{Unlike} \end{array} \right\}$ Signs give $\left\{ \begin{array}{l} + \\ - \end{array} \right\}$ in the Product.

Multiplicand		Multiplier		Product
That is $\left\{ \begin{array}{l} + \\ + \end{array} \right\}$	into	$\left\{ \begin{array}{l} + \\ + \end{array} \right\}$	gives	$\left\{ \begin{array}{l} + \\ - \end{array} \right\}$

For to Multiply an *Affirmative* by $\left\{ \begin{array}{l} \text{an Affirmative} \\ \text{a Negative} \end{array} \right\}$ Quantity, is ſo often to repeat the $\left\{ \begin{array}{l} \text{Affirmation} \\ \text{Negation} \end{array} \right\}$ of it; by *Def. of Mult.*

Therefore $\left\{ \begin{array}{l} + \\ - \end{array} \right\}$ into $+$, gives $\left\{ \begin{array}{l} + \\ - \end{array} \right\}$

But to Multiply a *Negative* by a *Negative* Quantity, is ſo many times to deny that *Negation*.

And ſo many times to deny the *Negation* of a *Thing*, is ſo many times to *Affirm* that *Thing*.

Therefore $-$ into $-$ gives $+$.

P R O B.

PROB. I.

To Multiply Simple Quantities.

RULE.

Join the Factors together, and prefix to them the Product of the Coefficients, if there be any, observing the foregoing Note.

EXAMPLES.

Mult.	$+a$	$-a$	$+a$	$-a$	$+a$	$-3ab$
By	$+b$	$-b$	$-b$	$+b$	$+3$	5
Prod.	$+ab$	$+ab$	$-ab$	$-ab$	$+3a$	$-15ab$

PROB. II.

To Multiply Compound Quantities,

RULE.

Each part of the Multiplier must be drawn into each part of the Multiplicand; by Lem. 2. Chap. 1. §. 1.

Examples, Of Compounds by Simple.

Mult.	$a + r$	$x - y$	$rr - 6$	$a - n + m$
By	n	x	3	$-y$
Prod.	$na + nr$	$xx - xy$	$3rr - 18$	$-ya + yn - ym$

Examples,

Examples, Of Compounds by Compounds.

Mult. $a + c$	$2x - a + 3$
By $a - c$	$x - 4$
$aa + ac$	$2xx - xa + 3x$
$-ac - cc$	$-8x + 4a - 12$
Prod. $aa - ac$	$2xx - xa - 5x + 4a - 12$

Mult. $4a^3 + 3a^2 - 2a + 1$
By $a^2 - 5a + 6$

Prod. $4a^5 - 17a^4 + 7a^3 + 29a^2 - 17a + 6$

Mult. $ZZZ - 2nZZ - nnn$
$+ a + aaa$
$- 2n^2a$

By $ZZZ + 2nZ + 4nn$
$- a - 4na$
$+ aa$

$Z^5 - 2nZ^4$	$- 4n^2Z^3$	$- n^3Z^2$	$- 2n^4Z$	$- 4n^5$
$+ a$	$+ 2na$	$+ a^3$	$+ 2na^2$	$+ 4n^2a^2$
$+ 2n$	$+ 2na$	$- 2n^2a$	$- 4n^3a$	$- 8n^4a$
$- a$	$- aa$	$- 8n^3$	$+ n^3a$	$+ 4n^4a$
	$+ 4n^2$	$+ 4n^2a$	$- a^4$	$- 4n^5a^2$
	$- 4na$	$+ 8n^2a$	$+ 2n^2a^2$	$+ 8n^3a^2$
	$+ aa$	$- 4na^2$		$- n^3a^2$
		$- 2na^2$		$+ a^5$
		$+ a^3$		$- 2n^2a^3$

Pr. Z^5	*	$- 9n^3ZZ$	$- 2n^4Z$	$- 4n^5$
		$+ 2a^3$	$+ 2na^2$	$+ 2n^2a^2$
		$+ 10n^2a$	$- 3n^3a$	$- 4n^4a$
		$- 6na^2$	$- a^4$	$- 4na^4$
			$+ 2n^2a^2$	$+ 7n^3a^2$
				$+ a^5$

SCHO.

S C H O L I U M 1.

Sometimes *Products* are express'd only by the Quantities to be *Multiplied* with the sign $\times$ between them.

Thus, The Product of $a+x$ by $e+z$, is $a+x \times e+z$.
And the Product of $a+x$ by $m-n+y$, and that Product by $e+z$

$$\text{Is } \overline{a+x} \times \overline{m-n+y} \times \overline{e+z}.$$

S C H O L I U M 2.

The Quantity produc'd by the *Multiplication* of *Two*, *Three*, &c. *Quantities*, is said to be of *Two*, *Three*, &c. *Dimensions*; and the *Quantities* thus *Multiplied*, are call'd *Roots*;

Thus $\begin{Bmatrix} a & b \\ a & b & c \end{Bmatrix}$ is a Quantity of $\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$ *Dimensions*, whose *Roots* are $\begin{Bmatrix} a, b \\ a, b, c. \\ \&c. \end{Bmatrix}$

But if the *Roots* are the same, then the *Quantities* produced are usually call'd *Powers*, as

$\begin{Bmatrix} a \\ a & a \\ a & a & a \\ \&c. \end{Bmatrix}$ is the $\begin{Bmatrix} 1\text{st.} \\ 2\text{d.} \\ 3\text{d.} \\ \&c. \end{Bmatrix}$ *Power* of a , and may be thus express'd $\begin{Bmatrix} a^1 \\ a^2 \\ a^3 \\ \&c. \end{Bmatrix}$

C H A P.

C H A P. V.

Division of Algebraic Integers.

General R U L E.

Set the Divisor under the Dividend, with a Line between them.

E X A M P L E S.

$$\begin{array}{r|l}
 \text{Divide} & a \mid nn+r \mid a+bc-dd \mid an-nx \\
 \text{By} & c \mid x-y \mid x+zz \mid a+u \\
 \hline
 \text{Quotient} & \frac{a}{c} \mid \frac{nn+r}{x-y} \mid \frac{a+bc-dd}{x+zz} \mid \frac{an-nx}{a+u}
 \end{array}$$

For by *Common Division*, in Dividing 6 by 2, 'tis the same thing whether 3, or $\frac{6}{2}$ be the *Quotient*.

But sometimes this Mark $\div$ is used as a sign of *Division*.

S C H O L I U M 1.

If a *Quantity* is found to be a *Common Multiplier* in both, it may be expunged from both; Observing that

$$\begin{array}{c}
 \text{Dividend} \\
 \left. \begin{array}{c} + \\ - \\ + \\ - \end{array} \right\} \begin{array}{c} + \\ - \\ - \\ + \end{array} \left. \begin{array}{c} + \\ - \\ - \\ + \end{array} \right\} \text{Quotient.} \\
 \text{Divisor}
 \end{array}$$

From the like Reason with that in *Multiplication*.

E X.

EXAMPLES.

$$\begin{array}{r|l}
 \text{Divide } +ac & -ac \\
 \text{By } +a & -a \\
 \hline
 \text{Quot. } +c & +c
 \end{array}
 \quad
 \begin{array}{r|l}
 +ac & -ac \\
 -a & -a \\
 \hline
 +c & +c
 \end{array}
 \quad
 \begin{array}{r|l}
 +ac & -ac \\
 -a & -a \\
 \hline
 +c & +c
 \end{array}
 \quad
 \begin{array}{r|l}
 +ac & -ac \\
 +a & +a \\
 \hline
 -c & -c
 \end{array}
 \quad
 \begin{array}{r|l}
 na+nc & \\
 n & \\
 \hline
 a+c &
 \end{array}$$

For $e \times a = ac$, and $a+c \times n = na+nc$. by Theor. Chap. 5. §. 1.

$$\begin{array}{r|l}
 \text{Divide } am+an & (a \mid \frac{ra-ac+rn-cn}{r-c} (a+n) \\
 \text{By } m+n & r-c
 \end{array}$$

SCHOLIUM 2.

In reducing these *Quotients* into lower Terms, it happens frequently, when the *Dividend* and *Divisor* are *Multinomial* Quantities, that there may be a Common Multiplier of both, which does not immediately appear; yet is found (as in Ordinary *Arithmetic*) by this

RULE.

Find one Member of the *Quotient*, by which Multiply the *Divisor*, Subtracting the Product from the *Dividend*, and by the Remainder seek another Member of the *Quotient*, and proceed thus, till the Operation be finished.

EXAMPLES.

$$\begin{array}{r}
 x-2 \quad x^3 - 2^3 \quad (x^2 + x2 + 2^2 \\
 \hline
 0 + x^2 2 - 2^3 \\
 \hline
 0 + x2^2 - 2^3 \\
 \hline
 0 \qquad 0
 \end{array}$$

The

The manner of Operation.

1. Find what Quantity Multiplied by x (the first Member of the Divisor) will give x^3 , (the first Member of the Dividend) and that must be x^2 ; whereby Multiply $x - z$, the Product $x^3 - x^2 z$ Subducted from the Dividend, leaves $x^2 z - z^3$.

2. Seek what Quantity Multiplied by x , gives $x^2 z$, that must be $z x$; which (as before) Multiplied and Subducted leaves $x z^3 - z^3$.

3. In the same manner, seek the third Member of the Quotient, which Multiplied and Subducted, leaves 0.

$$\begin{array}{r}
 x - 19) x^6 - 8x^4 - 124x^2 - 64 (x^4 + 8x^2 + 4 \\
 \underline{x^6 + 16x^4} \\
 - 8x^4 \\
 + 8x^4 - 128x^2 \\
 \underline{ - 4x^2} \\
 + 4x^2 - 64 \\
 \underline{ 0}
 \end{array}$$

$$\begin{array}{r}
 x^2 - n^2) x^6 + n^2 x^4 - n^4 x^2 - n^6 (x^4 + 2n^2 x^2 + n^4 \\
 \underline{-y^2 - 2y^2 + y^4 - 2n^4 y^2 - n^2 y^4} \\
 -n^2 x^4 \\
 \underline{-y^2} \\
 + 2n^2 x^4 - 2n^4 x^2 \\
 \underline{-y^2 - n^2 y^2} \\
 + y^4 \\
 + n^4 x^2 - n^6 \\
 \underline{+ n^2 y^2 - 2n^4 y^2 - n^2 y^4} \\
 0
 \end{array}$$

S C H O.

S C H O L I U M 3.

If the *Divisor* be not an even Part of the *Dividend*, the Operation may either be terminated by annexing to the Quotient, the Remainder set over the Divisor, with a Line drawn between them ; or else continued on in an *Infinite Series*.

E X A M P L E S.

$$\text{Divide } 9n^3y^4 \quad (= \frac{9n^3y^3}{2} \mid \frac{8n^3y^6}{4n^2y^2} (= \frac{2ny^5}{a}$$

$$2n) \quad \frac{8n^6 + 4n^2a^2y^2 - 3n^2a^4(4n^5 + 2na^2y^2) - \frac{3na^4}{2}}{0 \quad 0}$$

$$-2n) \quad \frac{10n^5 - 12n^3a^2 - 6a^2c^3 (-5n^4 + 6n^2a^2 + \frac{3a^2c^3}{n})}{0 \quad 0}$$

$$aa - cc) \quad aac \quad (c + \frac{c^3}{aa} + \frac{c^5}{a^4} + \frac{c^7}{a^6} + \frac{c^9}{a^8} + \&c.$$

$$\frac{aac - c^3}{c^3}$$

$$\frac{c^3 - \frac{c^5}{aa}}{\frac{c^5}{aa}}$$

$$\frac{\frac{c^5}{aa} - \frac{c^7}{a^4}}{\frac{c^7}{a^6}}$$

$$\frac{\frac{c^7}{a^6} - \frac{c^9}{a^8}}{\&c.}$$

$$aa-33) aaaa x(ax + \frac{3^2 x}{a} + \frac{3^4 x}{a^2 a} + \frac{3^6 x}{a^5} + \&c.$$

$$\begin{array}{r} aaaa x - a33 x \\ \hline a33 x \\ a33 x - \frac{3^2 x}{a} \\ \hline \frac{3^4 x}{a} \\ \frac{3^4 x}{a} - \frac{3^6 x}{a^2} \\ \hline \frac{3^6 x}{a^2} \\ \frac{3^6 x}{a^2} - \frac{3^8 x}{a^5} \\ \hline \frac{3^8 x}{a^5} \\ \&c. \end{array}$$

$$1-x) 1(1+x+xx+x^3+x^4, \&c,$$

$$\begin{array}{r} 1-x \\ \hline x \\ x - xx \\ \hline xx \\ xx - xxx \\ \hline xxx \\ xxx - x^4 \\ \hline x^4 \\ x^4 - x^5 \\ \hline x^5 \\ \&c. \end{array}$$

SCHOL.

S C H O L I U M 4.

In *Multiplication* and *Divison*, 'tis indifferent (as to Order) how the *Quantities* stand, so the Operations be Performed successively.

C H A P. VI.

Involution of Quantities.

D E F I N I T I O N.

A Quantity Multiplied into its self any Number of Times is said to be Involved; the Products arising are called Powers, and the Quantitiy so Multiplied a Root.

$$\text{Thus } \begin{cases} a \times a = a a \text{ or } a^2 \\ a \times a \times a = a a a \text{ or } a^3 \\ \text{\&c.} \end{cases}$$

Here *a* is the Root or First Power; a^2 , a^3 , &c. The Second, Third, &c. Powers of it.

S C H O L I U M.

Some of these Powers have borrowed their *Denominations* from *Local Extension*.

For a *Line* having but one *Dimension*, viz. *Length*, drawn into it self, produces a *Square-Plane*.

And that *Square* having two *Dimensions*, viz. *Length* and *Breadth*, drawn into it self, produces a *Cubed-Solid*.

This

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This Cube has three *Dimenſions*, viz. Length, Breadth, and *Thickneſs*; But the *Nature* and *Property* of *Space* admits of no other *Extenſion*.

Whence it follows, That the *Root* or *Fiſt Power* being taken as a *Side*, the *Second Power* will be a *Square*, the *Third* a *Cube*.

LEMMA.

Any Quantity (as a Root) Divided into Parts at Pleaſure, the Sum of the Product of thoſe Parts drawn into themſelves, and into each other any Number of Times, will be equal to that Quantity drawn into it ſelf the like Number of Times.

Thus, If the Root (*r*) be made a Binomial $a + e$.

Then, $r^2 = a^2 + 2ae + e^2$ } firſt } Pow-
And $r^3 = a^3 + 3a^2e + 3ae^2 + e^3$ } ſecond } er.
Or.

For,

$$\text{Firſt Power} = a + e = \text{Root}$$

$$\begin{array}{r} a^2 + 2ae \\ ae + e^2 \end{array}$$

$$\text{Second Power} = a^2 + 2ae + e^2 = \text{Square.}$$

$$\begin{array}{r} a^3 + 2a^2e + 4ae^2 + e^3 \\ a^2e + ae^2 + e^3 \end{array}$$

$$\text{Third Power} = a^3 + 3a^2e + 3ae^2 + e^3 = \text{Cube.}$$

The Method of Proceeding is the ſame in the Generation of Higher Powers, from any given Root, whether Binomial, or Multinomial.

COROLLARY.

Now from the Nature of the *Dimension* of each *Species*, their *Place* and *Value* in Numbers are soon Discovered.

As if the Root $\begin{cases} r = a + c \\ 24 = 20 + 4 \end{cases}$

Its Square is $\begin{cases} aa + 2ac + cc = r r. \\ 20 \times 20 + 2 \times 20 \times 4 + 4 \times 4 \\ i. e. 400 + 160 + 16 \end{cases} = 576$

Its Cube is $\begin{cases} aaa + 3aac + 3acc + ccc = r r r \\ 20 \times 20 \times 20 + 3 \times 20 \times 20 \times 4 + 3 \times 20 \times 4 \times 4 + 4 \times 4 \times 4 \\ i. e. 8000 + 4800 + 960 + 64 = 13824 \end{cases}$

Also for the *Square* of a *Binomial Number*.

Root $75 = 70 + 5 = a + c$

49..	4900	49..	a^2
.70.	.700	35.}	$2ac$
..25	..25	35.}	c^2
		25	

Square $5625 = 5625 = 5625 = a^2 + 2ac + c^2$

And for the *Cube*.

Root $75 = 70 + 5 = a + c$

343...	343000	aaa
735..	73500	3aac
525.	5250	3acc
125	125	ccc

Cube $421875 = 421875 = a^3 + 3a^2c + 3ac^2 + c^3$

In other Roots consisting of more Figures, the like Process is to be used.

1. Taking the Left-hand Figure for (*a*), and the next to it for (*e*); then by the *Theorem* for the *Power* set all the Parts in their due Places, and their Sum is the *Power* of those two Figures.

2. These two Figures being taken for (*a*), the next for (*e*), and Proceeding as before, you'll have the *Power* of the three Figures; and so on, till you have compleated the *Power* of the whole *Root*. Thus,

For the *Square* or second *Power* of a *Multisingmial Number*.

4 6 3 5 7	Root.		
16	..	= 40000 X 40000	= aa
48	..	= 40000 X 6000 X 2	= 2ae
36	..	= 6000 X 6000	= ee
2116	..	= 46000 X 46000	= aa
276	..	= 46000 X 300 X 2	= 2ae
9	..	= 300 X 300	= ee
214369	..	= 46300 X 46300	= aa
4630	..	= 46300 X 50 X 2	= 2ae
25	..	= 50 X 50	= ee
21483225	..	= 46350 X 46350	= aa
64890	..	= 46350 X 7 X 2	= 2ae
49	..	= 7 X 7	= ee
2148971449	Square.		

H

For

For the *Cube*, or *Third Power* of a *Multinomial Number*.

4 6 3			Root
64 ...	=	400 X 400 X 400	= a a a
288..	=	400 X 400 X 60 X 3	= 3 a a c
432.	=	400 X 60 X 60 X 3	= 3 a c c
216	=	60 X 60 X 60	= c c c
97336...	=	460 X 460 X 460	= a a a
19044..	=	460 X 460 X 3 X 3	= 3 a a c
1242.	=	460 X 3 X 3 X 3	= 3 a c c
27	=	3 X 3 X 3	= c c c
99252847			Cube

The like Method is to be observed in the Composition of other *Powers*, each according to the Nature of the *Theorem* thereto belonging.

S C H O L I U M.

This way of raising *Powers* from their *Component Parts*, rather leads to the Method of finding the *Root* of a given *Power*, than to find the *Power* of a given *Root*, which is soon done by the *Continual Multiplication* of that *Root*.

C H A P.

C H A P. VII.

Evolution of Quantities.

DEFINITION.

THE Resolution of Powers *into their*
Roots is called Evolution or the
Analysis of Powers.

S C H O L I U M.

And since any Root may be considered, as consisting of two Parts, one of which is supposed already known, and the other, tho' unknown, is yet discoverable by means of *Theorems* rais'd by the *Involution* of the *Binomial Root*; in which 'tis very easy to discern how each part of the Root is concern'd in the Power, and consequently how much is already known, and what remains farther to be enquired for.

Whence, the Method of *Extracting the Root* of any Power, may without difficulty be perform'd by observing only the *Constitution* of Powers, which plainly directs what manner of Operation each requires.

Now, the first thing to be done, is to distinguish the given Number into several Parts, by *Points* set over such *Places* as the *Index* of the Power directs.

Vir. That of *Squares, Cubes, &c.* into *Two's, Three's, &c.* beginning always from the *Place* of *Units*, and so towards the *Left-hand* in *Integers*, towards the *Right-hand* in *Decimals*; and there will be as many *Places* in the *Roots*, as there are *Points*.

The Method of *Extracting* the *Roots* of the *Second*
Power.

Literal E X A M P L E S.

I.
$$\begin{array}{r} aa + 2ac + cc \quad (a + c \\ \underline{aa} \end{array}$$

$$\begin{array}{r} 2a + c \quad \circ \quad \begin{array}{r} + 2ac + cc \\ + 2ac + cc \\ \hline \end{array} \\ \circ \quad \circ \end{array}$$

II.
$$\begin{array}{r} a^2 - 6an + 9n^2 + 2ax - 6nx + x^2 \quad (a - 3n + x \\ \underline{aa} \end{array}$$

$$\begin{array}{r} 2a - 3n \quad \circ \quad \begin{array}{r} - 6an + 9n^2 \\ - 6an + 9n^2 \\ \hline \end{array} \end{array}$$

$$\begin{array}{r} 2a - 6n + x \quad \circ \quad \begin{array}{r} + 2ax - 6nx + x^2 \\ + 2ax - 6nx + x^2 \\ \hline \end{array} \\ \circ \quad \circ \quad \circ \end{array}$$

III.
$$\begin{array}{r} a^4 + 4a^3c + 6a^2c^2 + 4ac^3 + c^4 \quad (a^2 + 2ac + c^2 \\ \underline{a^4} \end{array}$$

$$\begin{array}{r} 2a^2 + 2ac \quad \circ \quad \begin{array}{r} + 4a^3c + 6a^2c^2 \\ + 4a^3c + 6a^2c^2 \\ \hline \end{array} \end{array}$$

$$\begin{array}{r} 2a^2 + 4ac + c^2 \quad \circ \quad \begin{array}{r} + 2a^2c^2 + 4ac^3 + c^4 \\ + 2a^2c^2 + 4ac^3 + c^4 \\ \hline \end{array} \\ \circ \quad \circ \quad \circ \end{array}$$

Numeral

Numeral *E X A M P L E.*

Theorically thus.

$aa + 2ac + cc$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 178929(423 \end{smallmatrix}$
$aa = 4 \times 4 = 16 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 16 \dots \end{smallmatrix}$
$2a = 2 \times 4 = 8 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 8 \dots \end{smallmatrix}$
$2ac = 8 \times 2 = 16 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 16 \dots \end{smallmatrix}$
$cc = 2 \times 2 = 4 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 4 \dots \end{smallmatrix}$
$2a = 42 \times 2 = 84 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 84 \dots \end{smallmatrix}$
$2ac = 84 \times 3 = 252 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 252 \dots \end{smallmatrix}$
$cc = 3 \times 3 = 9 \dots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 9 \dots \end{smallmatrix}$
	000000

Practically thus.

$$\begin{array}{r}
 \begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ 178929(423 \end{smallmatrix} \\
 \hline
 82) \begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ .189 \dots \end{smallmatrix} \\
 \hline
 843) \begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ .2529 \end{smallmatrix} \\
 \hline
 0000
 \end{array}$$

The Method of *Extracting* the *Roots* of the *Third Power.*

Literal *E X A M P L E.*

$$\begin{array}{r}
 a^3 + 3a^2c + 3ac^2 + c^3 \quad (a+c \\
 \hline
 3a^3 + 3ac + c^3) 0 + 3a^2c + 3ac^2 + c^3 \\
 \hline
 + 3a^2c + 3ac^2 + c^3 \\
 \hline
 0 \quad 0 \quad 0
 \end{array}$$

Numeral

Numeral *E X A M P L E*.*Theorically* thus.

$a^3 + 3a^2e + 3ae^2 + e^3 =$	$75686967(423$
$a^3 = 4 \times 4 \times 4 =$	$64 \dots$
$3a^2 = 3 \times 4 \times 4 =$	$48) 11686 \dots (2 = e$
$3a^2e = 3 \times 4 \times 4 \times 2 =$	$96 \dots$
$3ae^2 = 3 \times 4 \times 2 \times 2 =$	$48 \dots$
$e^3 = 2 \times 2 \times 2 =$	$8 \dots$
$3a^2e + 3ae^2 + e^3 =$	$10088 \dots$
$3a^2 = 3 \times 42 \times 42 =$	$5292) 1598967 (3 = e$
$3a^2e = 3 \times 42 \times 42 \times 3 =$	$15876 \dots$
$3ae^2 = 3 \times 42 \times 3 \times 3 =$	$1134 \dots$
$e^3 = 3 \times 3 \times 3 =$	$27 \dots$
$3a^2e + 3ae^2 + e^3 =$	1598967
	0000000

Practically thus.

$75686967(423$
$48) 11686$
$96 \dots$
$48 \dots$
$8 \dots$
$5292) 1598967$
$15876 \dots$
$1134 \dots$
$27 \dots$
0000000

N O T E.

If the Number is not an exact *Square, Cube, &c.* Annex to the Remainder, *Two's, Three's, &c.* of Cyphers, so any desired Number of *Decimal Places* may be had in that *Manner*.

S E C T.

S E C T. III.

Of the Comparisons of Quantities.

C H A P. I.

Of Proportion.

D E F I N I T I O N I.

THE Relation of two Homogeneous Quantities one to another, may be considered, either,

1. By *how much* the one Exceeds the other, which is called their *Difference*.

Thus 5 exceeds 3 by the *Difference* 2.

2. Or *what Part or Parts* one is of another, which is call'd *Ratio*.

Thus, the *Ratio* of $\begin{cases} 6 \text{ to } 3 \text{ is } \frac{6}{3} = \frac{2}{1} \text{ or } \textit{Double} \\ 3 \text{ to } 6 \text{ is } \frac{3}{6} = \frac{1}{2} \text{ or } \textit{Subduple.} \end{cases}$

Note, That the Quantity Compared is called the *Antecedens*; and that to which it is Compared, is called the *Consequent*.

D E F I N I -

DEFINITION II.

When two $\left\{ \begin{array}{l} \text{Differences} \\ \text{Ratio's} \end{array} \right\}$ are equal, the Terms that Compose them are said to be $\left\{ \begin{array}{l} \text{Arithmetically} \\ \text{Geometrically} \end{array} \right\}$ Proportional.

S C H O L I U M I.

Suppose the Terms to be a . and b , their Difference d .

If a be the $\left\{ \begin{array}{l} \text{least} \\ \text{greatest} \end{array} \right\}$ Term, then $\left\{ \begin{array}{l} a + d \\ a - d \end{array} \right\} = b$.

Therefore in any Arithmetic Proportion when the Antecedent is $\left\{ \begin{array}{l} \text{less} \\ \text{greater} \end{array} \right\}$ than the Consequent, the Terms may be express'd by a and $\left\{ \begin{array}{l} a + d \\ a - d \end{array} \right\}$.

S C H O L I U M II.

Suppose a and b to be the Terms of any Ratio;

If a be the least Term

Put $r = \frac{b}{a}$, then $a r = b$ by Equal Mults.

But if b be the least Term.

Put $r = \frac{a}{b}$, then $b r = a$ by Equal Mults.

And $\frac{a}{r} = b$ by Equal Divisibn.

Therefore

Therefore, in any *Geometric Proportion*, when the *Antecedent* is $\left\{ \begin{smallmatrix} \text{less} \\ \text{greater} \end{smallmatrix} \right\}$ than the *Consequent*, the *Terms* may be express'd by $\left\{ \begin{smallmatrix} a \text{ and } ar \\ a \text{ and } \frac{a}{r} \end{smallmatrix} \right\}$.

DEFINITION III.

Those *Quantities* whose *Excess* or *Quotients* are the same, are call'd *Proportionals*.

CASE. I.

When of several *Quantities* the $\left\{ \begin{smallmatrix} \text{Difference} \\ \text{Quotient} \end{smallmatrix} \right\}$ of the 1st. and 2d. is the same with that of the 2d. and 3d. they are said to be in a *Continued* $\left\{ \begin{smallmatrix} \text{Arithmetic} \\ \text{Geometric} \end{smallmatrix} \right\}$ *Proportion*.

Thus, $\left\{ \begin{smallmatrix} a, a+d, a+2d, a+3d, a+4d, \\ a, a-d, a-2d, a-3d, a-4d, \end{smallmatrix} \right\}$ &c.
is a Series of *Continued Arithmetic Proportionals*, whose *Common Difference* is d .

And $\left\{ \begin{smallmatrix} a, ar, ar^2, ar^3, ar^4, ar^5 \\ a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}, \frac{a}{r^5} \end{smallmatrix} \right\}$ &c.

is a Series of *Continued Geometric Proportionals*, whose

Common Multiplier is $\left\{ \begin{smallmatrix} r \\ \frac{1}{r} \end{smallmatrix} \right\}$ or whose *Ratio* is that of $\left\{ \begin{smallmatrix} 1 \text{ to } r \\ r \text{ to } 1 \end{smallmatrix} \right\}$.

Note, That the Sign $::$ Signifies *Continued Proportion*.

CASE II.

When of several Quantities the $\left\{ \begin{array}{l} \text{Difference} \\ \text{Quotient} \end{array} \right\}$ of the 1st. and 2d. is the same with that of the 3d. and 4th. (and not of the 2d. and 3d.) they are said to be in a Discontinued $\left\{ \begin{array}{l} \text{Arithmetic} \\ \text{Geometric} \end{array} \right\}$ Proportion; such as,

$$1. \left\{ \begin{array}{l} a, a+d; c, c+d \\ a+d, a; c+d, c \end{array} \right\}$$

$$\text{For } \overline{a+d} - a = \overline{c+d} - c = d.$$

$$2. \left\{ \begin{array}{l} a, ar; c, cr; \\ ar, a; cr, c \end{array} \right\} \text{ for } \left\{ \begin{array}{l} \frac{a}{ar} = \frac{c}{cr} = \frac{1}{r} \\ \frac{ar}{a} = \frac{cr}{c} = \frac{r}{1} \end{array} \right.$$

CHAP. II.

The Chief Properties of Arithmetic Proportion.

THEOREM I.

In any Number of Continued Arithmetic Proportionals:

1. If the Number of Terms be even;
The Sum of the Extreams, and that of every two equally distant from them, are equal.
2. If the Number of Terms be odd;
Then those Sums are each equal to the Double of the Middle Term.

Thus

Thus in, $a, a+d, a+2d, a+3d, a+4d$.
 'Tis evident that $a + a + 4d = a + d + a + 3d = 2a + 4d$.

C O R O L L A R Y I.

Therefore, *If three Quantities are in a Continued Arithmetic Proportion;*

The Sum of the Extreams is equal to the Double of the Middle Term.

P R O B L. I.

To find any Number (n) of Arithmetical Mean Proportionals, between any two given Quantities, a , and e .

Since, $a, a+d, a+2d, a+3d, \&c. a + \overline{n+1} \times d = e$.

Therof. $d \times \overline{n+1} = e - a$; and $d = \frac{e - a}{n+1}$.

Or putting $x = \text{First mean} = \text{Second Term}$.

Then $a, x, 2x-a, 3x-2a, 4x-3a, \&c. x \times \overline{n+1} - na = e$

Therefore $x \times \overline{n+1} = e + na$, and $x = \frac{e + na}{n+1}$.

Now having either the *Difference*, or the *First Mean*, the rest are soon found.

P R O B. II.

To find a 3d. 4th. 5th. or n th. Arithmetic Proportional to any two given Quantities, a and x .

'Tis evident $a + \overline{n-1} \times d$, or $x \times \overline{n-1} - a \times \overline{n-2}$ is the *Proportional* required by the preceding Problem.

COROLLARY II.

If in a Rank of *Continued Arithmetical Proportionals*, there be taken any *Series* of Equidistant Terms, That *Series* will be also *Proportional*.

For if, $a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, \&c.$
 Then $\begin{cases} a+d, & a+3d, & a+5d, & \&c. \\ a, & a+2d, & a+4d, & a+6d, & \&c. \\ \&c. & & a+3d, & a+5d, & \&c. \end{cases}$

THEOREM. 2.

If 4 Quantities be *Arithmetically Proportional*; The Sum of the Extreams is equal to the Sum of the Means.

That is, if $\begin{cases} a, a+d, a+2d, a+3d, \text{ Continued.} \\ a, a+d; \quad c, c+d, \text{ Discontinued.} \end{cases}$

'Tis plain $\begin{cases} a + \overline{a+3d} = \overline{a+d} + \overline{a+2d} = 2a + 3d \\ a + \overline{c+d} = \overline{a+d} + c. \end{cases}$

CHAPTER III.

The Chief Properties of Geometric Proportion.

THEOREM 1.

In any Number of *Continued Geometric Proportionals*:

1. If the Number of Terms be even;

The *Products* of the Extreams, and that of every two Terms, equally distant from them are equal.

2. If the Number of Terms be odd;

Then those *Products* which are each equal to the Square of the Middle Term.

Thus,

Thus, in

$$a, ar, ar^2, ar^3, ar^4, ar^5, ar^6,$$

'Tis evident that,

$$a \times ar^6 = ar \times ar^5 = ar^2 \times ar^4 = ar^3 \times ar^3 = a^2 r^6.$$

C O R O L L A R Y I.

Hence, If three Quantities are in a Continual Geometric Proportion;

The Product of the Extreams is equal to the Square of the Middle Term.

Thus in, a, ar, ar^2 .

'Tis Plain $a \times ar^2 = ar \times ar = a^2 \times r^2$.

P R O B L E M I.

To find any Number (n) of Geometrical Means between any two given Quantities, a and c .

Since, a, ar, ar^2, ar^3 , &c. $ar^{n+1} = c$

Therefore, $r^{n+1} = \frac{c}{a}$, And $r = \sqrt[n+1]{\frac{c}{a}} = \text{Ratio}.$

Or, putting $x = \text{First Mean} = \text{Second Term}.$

Then, $a, x, \frac{x^2}{a}, \frac{x^3}{a^2}, \frac{x^4}{a^3}$, &c. $\frac{x^{n+1}}{a^n} = c$

Therefore, $x^{n+1} = a^n c$, And $x = \sqrt[n+1]{a^n c}.$

Now, having found either the Ratio or First Mean, the rest are soon got.

P R O B.

P R O B L E M II.

Given, any two Quantities, a and x .

Required, a 3d, 4th, 5th, or n th, Geometric Proportion.

'Tis evident from the Preceding Problem, that $a \times r^{n-1}$, or $\frac{x^{n-1}}{a^{n-2}}$ is the Proportional required.

C O R O L L A R Y 2.

If out of a Rank of Continued Geometric Proportionals, there be taken any Series of Equidistant Terms, that Series shall be also Proportional.

Thus,

$$\begin{array}{l} \text{If, } a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, ar^7, ar^8 \\ \text{Then } \left\{ \begin{array}{l} a, ar^2, ar^4, ar^6, ar^8 \\ ar, ar^3, ar^5, ar^7 \\ a, ar^3, ar^5, ar^7 \\ a, ar^4, ar^6, ar^8 \end{array} \right\} \&c. \end{array}$$

T H E O R E M 2.

If 4 Quantities be Geometrically Proportional;
The Product of the Extreams, is equal to that of the Means.

That is, if $\left\{ \begin{array}{l} a, ar, ar^2, ar^3, \text{ Continued} \\ a, ar, c, cr \text{ Discontinued} \end{array} \right.$

'Tis plain that $\left\{ \begin{array}{l} a \times ar^3 = ar \times ar^2 (= a^2 r^3) \\ a \times cr = ar \times c (= acr = arc) \end{array} \right.$

Or,

Or, if $3:5::2 \times 3:2 \times 5$
i. e. $3:5::6:10$

Then $3 \times 10 = 5 \times 6$
i. e. $3 \times 2 \times 5 = 5 \times 2 \times 3 \} = 30.$

C O N V E R S E.

If 2 Products arising from the Multiplication of two Quantities be equal;

Those 4 Quantities will be Proportional.

C A S E I.

In any two Products, if the Factors of the one be made the 1st. Antecedent, and the 2d. Consequent; those of the other, the 1st. Consequent, and 2d. Antecedent, then the Terms are said to be Directly Proportional.

That is, if $A b = a B.$

Then $\left\{ \begin{array}{l} A : B :: a : b, \text{ or } A : a :: B : b. \\ B : A :: b : a, \text{ or } B : b :: A : a. \\ a : b :: A : B, \text{ or } a : A :: b : B. \\ b : a :: B : A, \text{ or } b : B :: a : A. \end{array} \right.$

S C H O L I U M.

If in 4 Quantities directly Proportional ($A:a::B:b$) any Three be given, the Fourth is readily found.

For $A b = B a$, i. e. 1st. $\times$ 4th. = 2d. $\times$ 3d. by this Theorem.

Therefore

$$\begin{array}{l} \text{Theref.} \\ \text{by equal} \\ \text{Division.} \end{array} \left\{ \begin{array}{l} 1^{\text{st}}. = \frac{2^{\text{d}} \times 3^{\text{d}}}{4^{\text{th}}} = 2^{\text{d}}. \times \frac{3^{\text{d}}}{4^{\text{th}}} = 3^{\text{d}}. \times \frac{2^{\text{d}}}{4^{\text{th}}} \\ 2^{\text{d}}. = \frac{1^{\text{st}} \times 4^{\text{th}}}{3^{\text{d}}} = 1^{\text{st}}. \times \frac{4^{\text{th}}}{3^{\text{d}}} = 4^{\text{th}}. \times \frac{1^{\text{st}}}{3^{\text{d}}} \\ 3^{\text{d}}. = \frac{1^{\text{st}} \times 4^{\text{th}}}{2^{\text{d}}} = 1^{\text{st}}. \times \frac{4^{\text{th}}}{2^{\text{d}}} = 4^{\text{th}}. \times \frac{1^{\text{st}}}{2^{\text{d}}} \\ 4^{\text{th}}. = \frac{2^{\text{d}} \times 3^{\text{d}}}{1^{\text{st}}} = 2^{\text{d}}. \times \frac{3^{\text{d}}}{1^{\text{st}}} = 3^{\text{d}}. \times \frac{2^{\text{d}}}{1^{\text{st}}} \end{array} \right.$$

NOTE.

That $\frac{Ba}{A}$ is a 4th Proportional to A, B, and a, may be proved also thus;

For since $A : Ba :: 1 : \frac{Ba}{A}$ by Def. of Division.

And $A \times \frac{Ba}{A} (= Ba \times 1) = Ba$ by this Theor.

Therefore, $A : B :: a : \frac{Ba}{A}$, by its Converse, q. c. d.

CASE II.

But if the Factors of the one be made the 1st. Antecedent, and 1st. Consequent; those of the other, the 2^d. Antecedent, and 2^d. Consequent, then the Terms are said to be Reciprocally Proportional.

That is, if $Ab = aB$.

$$\text{Then } \left\{ \begin{array}{l} A : b :: B : a, \text{ or } A : b :: a : B. \\ B : a :: A : b, \text{ or } B : a :: b : A. \\ a : B :: b : A, \text{ or } a : B :: A : b. \\ b : A :: a : B, \text{ or } b : A :: B : a. \end{array} \right.$$

S C H O.

S C H O L I U M.

If in 4 Quantities, *Reciprocally Proportional* ($A : b :: a : B$) any Three be given, the 4th is eaſily found.

For $A b = a B$, i. e. $1^{\text{ft}} \times 2^{\text{d}} = 3^{\text{d}} \times 4^{\text{th}}$ by the Def. of *Recipr. Proportion*.

$$\begin{array}{l} \text{Theref.} \\ \text{by equal} \\ \text{Diviſion,} \end{array} \left\{ \begin{array}{l} 1^{\text{ft}} = \frac{3^{\text{d}} \times 4^{\text{th}}}{2^{\text{d}}} = 3^{\text{d}} \times \frac{4^{\text{th}}}{2^{\text{d}}} = 4^{\text{th}} \times \frac{3^{\text{d}}}{2^{\text{d}}} \\ 2^{\text{d}} = \frac{3^{\text{d}} \times 4^{\text{th}}}{1^{\text{ft}}} = 3^{\text{d}} \times \frac{4^{\text{th}}}{1^{\text{ft}}} = 4^{\text{th}} \times \frac{3^{\text{d}}}{1^{\text{ft}}} \\ 3^{\text{d}} = \frac{1^{\text{ft}} \times 2^{\text{d}}}{4^{\text{th}}} = 1^{\text{ft}} \times \frac{2^{\text{d}}}{4^{\text{th}}} = 2^{\text{d}} \times \frac{1^{\text{ft}}}{4^{\text{th}}} \\ 4^{\text{th}} = \frac{1^{\text{ft}} \times 2^{\text{d}}}{3^{\text{d}}} = 1^{\text{ft}} \times \frac{2^{\text{d}}}{3^{\text{d}}} = 2^{\text{d}} \times \frac{1^{\text{ft}}}{3^{\text{d}}} \end{array} \right.$$

C O R O L L A R Y 2.

In Ranks of *Similar Proportions*, the Sums or Difference of the Corresponding Terms ſhall be *Proportional*.

$$\text{That is, if } \left\{ \begin{array}{l} a : ar :: e : er \\ x : xr :: r : rr \\ \text{etc.} \end{array} \right.$$

$$\text{Then } a \pm x : ar \pm xr :: e \pm r : er \pm rr.$$

C O R O L L A R Y 3.

In two Ranks of *Proportionals*, the Products or Quotients of the Corresponding Terms will be *Proportional*.

$$\text{That is, if } \left\{ \begin{array}{l} a : ar :: e : er \\ x : xs :: r : rs \end{array} \right.$$

K

Then

$$\text{Then } \left\{ \begin{array}{l} ax : ar :: ex : er \\ \frac{a}{x} : \frac{ar}{xs} :: \frac{e}{r} : \frac{er}{rs} \end{array} \right.$$

COROLLARY 4.

The Powers of Proportionals, are also Proportional.

That is,

<i>Continued</i>	<i>Discontinued</i>
If a, ar, ar^2, ar^3 &c.	a, ar, e, er Roots.
Then $\left\{ \begin{array}{l} a^2, a^2 r^2, a^2 r^4, a^2 r^6 \\ a^3, a^3 r^3, a^3 r^6, a^3 r^9 \end{array} \right\}$ &c.	$\left\{ \begin{array}{l} a^2, a^2 r^2; e^2, e^2 r^2 \\ a^3, a^3 r^3; e^3, e^3 r^3 \end{array} \right\}$ Squares. Cubes.

COROLLARY 5.

Products having the same Common Factor are as the other Factors.

$$\text{That is, } \left\{ \begin{array}{l} an : en :: a : e \\ anm : enm :: a : e \end{array} \right\} \&c.$$

For $an \times e = en \times a$; and $anm \times e = enm \times a$, &c.

SCHOLIUM I.

Hence, The Product of any two Quantities a and e , is a mean Proportional between the Squares of those Quantities.

$$\text{Thus } a : a :: ae : ae :: e : e.$$

For P. Means = P. Extremes.

SCHO-

SCHOLIUM 2.

Therefore, between a^n and a^1 , there are $n-1$ Mean Proportionals Composed of two Ranks of Powers,

The $\left\{ \begin{array}{l} \text{one In-} \\ \text{other De-} \end{array} \right\}$ creasing, from $\left\{ \begin{array}{l} a^{n-1} \text{ to } a^{n-n} \\ a^{n-n} \text{ to } a^{n-1} \end{array} \right\}$.

SCHOLIUM 3.

If Unit be put for the 1st. Term, the Root for the 2d. and its Consequent Powers for the 3d. 4th, &c. Terms (in order), they will be a Series of Geometric Proportionals, whose Common Ratio is the Root of Second Term; And their Exponents will be a Series of Arithmetic Proportionals, (expressing the Dimension of each Term, or its Distance from Unity) either Affirmative or Negative, according as the Root is More or Less than Unit.

Thus $\left\{ \begin{array}{l} +4, +3, +2, +1, 0, -1, -2, -3, -4, \text{ \&c.} \end{array} \right\}$ Expon. Powers.

But $\frac{a^4}{1}, \frac{a^3}{1}, \frac{a^2}{1}, \frac{a}{1}, 1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}, \dots$

Therefore, $a^{-n} = \frac{1}{a^n}$.

Hence it is, that a Quantity drawn into it self any Number of Times, that Number more one is the Exponent thereof.

Therefore, to $\left\{ \begin{array}{l} \text{Double} \\ \text{Triple} \\ \text{\&c.} \end{array} \right\}$ the Index of any $\left\{ \begin{array}{l} \text{Square} \\ \text{Cube} \\ \text{\&c.} \end{array} \right\}$ that Power, is to $\left\{ \begin{array}{l} \text{Square} \\ \text{Cube} \\ \text{\&c.} \end{array} \right\}$ Pow-er.

Conseq. $\left\{ \begin{array}{l} \frac{1}{2} \\ \frac{1}{3} \\ \text{\&c.} \end{array} \right\}$ of the Index of $\left\{ \begin{array}{l} \text{Square} \\ \text{Cube} \\ \text{\&c.} \end{array} \right\}$ any Power, is to $\left\{ \begin{array}{l} \text{Square} \\ \text{Cube} \\ \text{\&c.} \end{array} \right\}$ Root of that to take $\left\{ \begin{array}{l} \text{Square} \\ \text{Cube} \\ \text{\&c.} \end{array} \right\}$ Extract the Power.

Therefore, $\sqrt[n]{a^n}$ is more Naturally express'd by $a^{\frac{n}{n}}$.

Or, Because the *Root* is the 1st. *Geometrical Mean* between 1 and the *Power*;

But $\sqrt[r]{a^n}$ is the 1st. *Geometrical Mean* between the Powers 1 and a^n . by *Probl. 1. Cb. 3.*

And $\frac{n}{r}$ is the 1st. *Arithmetical Mean* between their Exponents 0 and n . by *Probl. 1. Cb. 2.*

Therefore $\frac{n}{r}$ is the *Exponent* of the *Power*.

That is, $\sqrt[r]{a^n} (= a^{n|\frac{1}{r}}) = a^{\frac{n}{r}}$.

Also, $\frac{1}{\sqrt[r]{a^n}} (= \frac{1}{a^{n|\frac{1}{r}}}) = a^{-\frac{n}{r}}$.

And $\frac{a^n}{\sqrt[r]{a^n}} = \frac{a^n}{a^{n|\frac{1}{r}}} = a^n \times \frac{1}{a^{n|\frac{1}{r}}} = a^{n-\frac{n}{r}} = a^{\frac{nr-n}{r}}$

For the same *Reason*.

COROLLARY 6.

If $A^n : a^n :: B : b$, then $A^m b = a^n B$
 Theref. $A : a :: B^n : b^m$, for $A^m b^m = a^n B^n$.

COROLLARY 7.

If $a : ar :: c : cr$

Then 1. $na : ar :: nc : cr$

2. $na : ar :: c : \frac{cr}{n}$

3. $a : \frac{ar}{n} :: nc : cr$.

COROLLARY 8.

If $A : a :: B : b$

i.e. $A : Ar :: B : Br$.

Then

Then 1. $AA : aa :: AB : ab.$

2. $Aa : Ab :: Ab : Bb.$

3. $AA : aB :: AB : Bb.$

4. $AB : Aa :: BB : Ab.$

C O R O L L A R Y 9.

If $\begin{cases} a : ar :: e : er. \\ x : xr :: e : er. \end{cases}$

Then $a \pm x : ar \pm xr :: e : er.$

C O R O L L A R Y 10.

If $\begin{cases} A : B :: a : b \\ C : B :: a : d \end{cases}$ then $A : C :: d : b.$

For $Ab (=Ba) = Cd$, by this *Theor.*

Therefore $A : C :: d : b.$ by its *Converse*.

C O R O L L A R Y 11.

In $\begin{cases} A, B, C \\ a, b, c \end{cases}$ if $\begin{cases} A : B :: b : c \\ B : C :: a : b \end{cases}$

Then $A : C :: a : c.$

For $Ac (=Bb) = Ca$, by this *Theorem*.

Therefore $A : C :: a : c$ by its *Converse*.

C O R O L L A R Y 12.

If $a : ar :: e : er.$

Then $\begin{cases} a : ar \\ e : er \end{cases} :: a \pm e : ar \pm er.$

C O R O L L A R Y 13.

In any Number of *Geometric Proportionals*;

As one Antecedent is to its Consequent;

So is the Sum of the Antecedents, to the Sum of the Consequents.

i. e.

i. e. If $\begin{cases} a, ar, ar^2, ar^3, \&c. & \text{Continued,} \\ a, ar, c, cr, \&c. & \text{Discontinued,} \end{cases}$

Then,

$$a : ar :: a + ar + ar^2 : \overbrace{a + ar + ar^2}^{a + ar + ar^2 \times r}$$

$$a : ar :: a + c : \overbrace{a + c}^{a + c \times r}$$

COROLLARY 14.

If $A : a :: B : b$. Then

1. Alternately, $A : B :: a : b$.
2. Inversely, $a : A :: b : B$.
3. Compound. $A + a : a :: B + b : b$.
4. Conversely, $A : A \pm a :: B : B \pm b$.
5. Dividedly, $A - a : a :: B - b : b$.
6. Mixtly, $A + a : A - a :: B + b : B - b$.

All evidently *Proportional*, since the *Products* of their *Extremes* and *Means* are the *same* with those of the given *Proportion*.

SCHOLIUM.

Proportions *Compounded*, *Converted*, *Divided* and *Mixt*, are also *Proportionals* when *Alterned* and *Inverted*.

COROLLARY 15.

$$\text{If } \begin{cases} a : ar :: c : cr \\ c : cr :: u : ur \end{cases}$$

Then $a : ar :: u : ur$.

SCHO-

SCHOLIUM.

In $\left\{ \begin{array}{l} A, B, C, \\ a, b, c, \end{array} \right\}$ if $\left\{ \begin{array}{l} A : B :: a : b \\ B : C :: b : c \end{array} \right\}$

Then $A : C :: a : c$.

For $A : a :: B : b$ } by *Altern.*
And $B : b :: C : c$ }

Therefore $A : a :: C : c$ by this *Cor.*

And $A : C :: a : c$ by *Altern.*

COROLLARY 16.

If $\left\{ \begin{array}{l} b : ar :: u : cr \\ a : ar :: u : ur \end{array} \right\}$

Then $c : cr :: u : ur$.

DEFINITION IV.

THE Product of the like Terms of any Ratio is called a Compound Ratio or a Ratio compounded of those Ratio's.

Thus, the Ratio $\frac{ac}{mn}$ is Compounded of $\frac{a}{m}$ and $\frac{c}{n}$, or of $\frac{a}{n}$ and $\frac{c}{m}$.

SCHOLIUM.

Whence, Compound Ratio's are produced by the Multiplication of the Ratio's compounding them;

That is, $\frac{ac}{mn} = \frac{a}{m} \times \frac{c}{n}$.

For

For $mnac = amnc$ by Schol. 4. Ch. 5. §. 2.

And $mn : an :: mc : ac$ by Conv. Theor. 2.

Theref. $\frac{mn}{mn} : \frac{an}{mn} :: \frac{mc}{mn} : \frac{ac}{mn}$ by equal Divis.

i. e. $1 : \frac{a}{m} :: \frac{c}{n} : \frac{ac}{mn}$

Conseq. $\frac{ac}{mn} = \frac{a}{m} \times \frac{c}{n}$ by Theor. 2.

DEFINITION V.

Similar Products are those whose Corresponding Factors are Proportional.

THEOREM 3.

All Similar Products are as the Terms of the Ratio of the Corresponding Factors, rais'd into the Product's Dimension.

Suppose ABC , &c. and abc , &c. the Products, whose Number of Dimension is n .

The Similar Factors being as r to s .

That is $\left\{ \begin{array}{l} A : a :: r : s \\ B : b :: r : s \\ C : c :: r : s \\ \text{\&c.} \end{array} \right.$

Theref. ABC , &c. : abc , &c. : $r^n : s^n$. by Cor. 3. Theor. 2.

COROLLARY 1.

The Ratio of any two Products is Composed of the Ratios of the Factors.

C O-

C O R O L L A R Y 2.

And all *Similar Powers* are in a *Ratio* Compounded of their *Roots* rais'd into the *Index* of these *Powers*.

C O R O L L A R Y 3.

Also *Ratio's* Compounded of equal ones are equal to one another.

Thus, if $\frac{a}{m} = \frac{e}{n}$, and $\frac{x}{y} = \frac{u}{z}$, then $\frac{ax}{my} = \frac{eu}{nz}$.

T H E O R E M 4.

In any given Quantities, (a, b, c, d, e,) the *Ratio* of the *Extreams* is *Compounded* of all the *Intermediate ones*.

That is, $\frac{a}{e} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} = \frac{a b c d}{b c d e}$.

C O R O L L A R Y.

Therefore, If $\left. \begin{array}{l} A : B :: a : b \\ B : C :: b : c \\ C : D :: c : d \\ \text{\&c.} \end{array} \right\}$ Then $A : D :: a : d$.

For $\frac{A}{B} = \frac{a}{b}$, $\frac{B}{C} = \frac{b}{c}$, $\frac{C}{D} = \frac{c}{d}$, by *Def.* of *Proport.*

And $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$.

That is, $\frac{A}{D} = \frac{a}{d}$, by this *Theor.*

Conseq. $A : D :: a : d$. by *Def.* of *Proport.*

LI

DEFI.

DEFINITION VI.

ALL Ratio's Compound of 2, 3, &c. Equal ones, are called the Duplicate, Triplicate, &c. of any one of those Ratio's.

Thus, if $\frac{a}{m} = \frac{c}{n}$, then $\frac{ac}{mn} = \frac{aa}{mm} = \frac{cc}{nn}$ is Duplicate of $\frac{a}{m}$, or $\frac{c}{n}$.

COROLLARY.

Therefore, In any Number of Continued Geometric Proportionals,

1, a , a^2 , a^3 , a^4 , a^5 , a^6 , &c.

The Ratio of the $\left\{ \begin{smallmatrix} 3d. \\ 4th. \\ 5th. \end{smallmatrix} \right\}$ 1st. to the $\left\{ \begin{smallmatrix} Duplicate \\ Triplicate \\ Quadrupl. \end{smallmatrix} \right\}$ is of that of the 1st. to the 2d. &c.

And Consequently,

The Ratio of the $\left\{ \begin{smallmatrix} 3d. \\ 4th. \\ 5th. \end{smallmatrix} \right\}$ 1st. to the $\left\{ \begin{smallmatrix} Duplicate \\ Triplicate \\ Quadrupl. \end{smallmatrix} \right\}$ is of that of the 2d. to the 1st.

SCHOLIUM.

In any Series of Geometric Proportionals;

a , ar , ar^2 , ar^3 , ar^4 , ar^5 , ar^6 , &c.

As $a^n : \overline{ar}^n :: a : a \times r^n$, that is,

As

As any *Power* of the 1st. is to the like of the 2d.

So is the 1st. to the $n + 1$ Term :

Or, So is the 1st. 2d. 3d. &c. to the n th. Term from it.

LEMMA 1.

To Multiply a Ratio by any Quantity, is to Multiply the Antecedent of that Ratio by that Quantity.

$$\text{That is, } m \times \frac{A}{B} = \frac{mA}{B}.$$

$$\text{For } B : A :: m : \frac{mA}{B} \text{ by Sch.to Conv. Theor. 2.}$$

$$\text{And } B : A :: 1 : \frac{A}{B} \text{ by Def. of Divis.}$$

$$\text{Theref. } 1 : \frac{A}{B} :: m : \frac{mA}{B} \text{ by Cor. 16. Theor. 2.}$$

$$\text{But } 1 : \frac{A}{B} :: m : \frac{A}{B} \times m, \text{ by Def. of Mult.}$$

$$\text{Conseq. } \frac{A}{B} \times m = \frac{mA}{B}.$$

THEOREM. 5.

Ratio's having the same Consequents are Directly as their Antecedents.

$$\text{That is, } \frac{A}{B} : \frac{a}{B} :: A : a$$

$$\left. \begin{array}{l} \text{For } a \times \frac{A}{B} = \frac{aA}{B}, \text{ and } A \times \frac{a}{B} = \frac{Aa}{B} \\ \text{But } \frac{aA}{B} = \frac{Aa}{B}, \text{ or } a \times \frac{A}{B} = A \times \frac{a}{B} \end{array} \right\} \text{by Lem. 1.}$$

$$\text{Theref. } \frac{A}{B} : \frac{a}{B} :: A : a. \text{ by Conv. Theor. 2.}$$

L 2

LEM.

L E M M A 2.

If a Quantity Multiplies another, and the Product be Divided by the same, the Quotient will be the Quantity Multiplied.

$$\text{That is, } a = \frac{an}{n} = \frac{anr}{nr}, \&c.$$

For $1 : a :: n : an$. by *Def. of Mult.*

And $n : an :: 1 : \frac{an}{n}$ by *Def. of Divis.*

Therof. $1 : a :: 1 : \frac{an}{n}$ by *Cor. 15. Theor. 2.*

Conseq. $a = \frac{an}{n}$, &c. by *Def. of Proport.*

C O R O L L A R Y.

Therefore If the Terms of a Ratio be Multiplied or Divided by the same Quantity, the Ratio will be the same.

$$\text{That is, } \frac{x}{n} = \frac{xr}{nr}, \&c.$$

For putting $an = x$, then $\frac{an}{n} = \frac{x}{n}$, and $\frac{anr}{nr} = \frac{xr}{nr}$

But $\frac{an}{n} (= a) = \frac{anr}{nr}$ by this *Lemma*.

Therefore $\frac{x}{n} = \frac{xr}{nr}$, &c. by *Substitution*.

T H E

THEOREM 6.

Ratio's having the ſame Antecedents are Reciprocaly as their Conſequents.

That is $\frac{A}{B} : \frac{A}{C} :: C : B$.

For $\frac{A}{B} = \frac{AC}{BC}$, and $\frac{A}{C} = \frac{BA}{BC}$ by prece^d. Cor.

But $\frac{AC}{BC} : \frac{BA}{BC} :: AC : BA$, by Theor. 5.

That is, $\frac{A}{B} : \frac{A}{C} :: C : B$.

THEOREM 7.

If $A : B :: a : b$

Then $\left\{ \begin{array}{l} nA : B :: na : b \\ A : nB :: a : nb \end{array} \right\}$ and $\left\{ \begin{array}{l} \frac{A}{n} : B :: \frac{a}{n} : b \\ A : \frac{B}{n} :: a : \frac{b}{n} \end{array} \right\}$

1. For $A : a :: B : b$, by Altern.
And $nA : na :: A : a$, by Cor. to. Lem. 2.
Theref, $nA : B :: na : b$, by Altern.

2. Alſo $\frac{A}{n} : \frac{a}{n} :: A : a$ by Theor. 5.

Theref. $\frac{A}{n} : B :: \frac{a}{n} : b$, by Altern.

THE-

THEOREM 8.

Quantities Proportional to their Differences are Continually Proportional.

For if $a : a - b :: b : b - c$

Then $a - \overline{a - b} : a - b :: b - \overline{b - c} : b - c$, by *Divid.*

That is $b : a - b :: c : b - c$

And $b : c :: a - b : b - c$ } by *Altern.* of $\left\{ \begin{array}{l} \text{Last} \\ \text{First} \end{array} \right\}$ } *&c.*

But $a : b :: a - b : b - c$

Theref. $a : b :: b : c$. by *Cor. 16. Theor. 2.*

THEOREM 9.

If Quantities are Continually Proportional, their Sums or Differences are also in Continual Proportion.

For if $a : b :: b : c :: c : d :: d : e$, &c.

Then $a : a \pm b :: b : b \pm c :: c : c \pm d :: d : d \pm e$.

And $\left\{ \begin{array}{l} a : b :: a \pm b : b \pm c \\ b : c :: b \pm c : c \pm d \end{array} \right\}$ by *Altern.* &c.

But $a : b :: b : c :: c : d$, &c. by *Sup.*

Theref. $a \pm b : b \pm c :: b \pm c : c \pm d$, &c.

THEOREM 10.

If $A : B :: a : b$; then $A : \overline{AB}^{\frac{1}{2}} :: a : \overline{ab}^{\frac{1}{2}}$.

For $AA : AB :: a : b$ }

And $AA : AB :: aa : ab$ } by *Cor. Lem. 2.*

Theref. $A : \overline{AB}^{\frac{1}{2}} :: a : \overline{ab}^{\frac{1}{2}}$, by *Evolution.*

CHAP.

C H A P. IV.

Of Harmonic and Contra-Harmonic Proportions.

DEFINITION I.

WHen 3 Terms are so disposed, that
the
1st. ∞ 2d. : 2d. ∞ 3d. :: 1st. : 3d.
they are said to be Harmonically Proportional.

Thus, 10, 15, 30, are Harmonic. Proportional.

For $10 \infty 15 : 15 \infty 30 :: 10 : 30$.

And $b^2 - bn$, $b^2 - n^2$, $b^2 + bn$, make an Harm. Prop.

For $bn - n^2 : bn + n^2 :: b^2 - bn : b^2 + bn$.

Also, 12, 6, 4, are Harmon. Proport.

For $12 - 6 : 6 - 4 :: 12 : 4$.

So $b^2 + 3bn + 2n^2$, $b^2 + 2bn$, $b^2 + bn$, are Har. Prop.

For $bn + 2n^2 : bn :: b^2 + 3bn + 2n^2 : b^2 + bn$.

C O R O L L A R Y.

Whence, If the 2 first Terms of an Harmonic Proportion be given, the 3d. is readily found.

For if a , b , c , be Harmonically Proportional.

Then, $a - b : b - c :: a : c$, and $ac - bc = ab - ac$.

Theref. $ab = 2a - b \times c$, and $bc = 2c - b \times a$.

Conseq. $c = \frac{ab}{2a - b}$, and $a = \frac{bc}{2c - b}$.

DEFI-

DEFINITION II.

When 4 Terms are so disposed, that
the

$$1^{\text{st}}. \infty 2^{\text{d}}. : 3^{\text{d}}. \infty 4^{\text{th}}. :: 1^{\text{st}}. : 4^{\text{th}}.$$

they are also Harmonically Proportional.

$$\text{As } 10, 16, 24, 60; \text{ For } 10 \infty 16 : 24 \infty 60 :: 10 : 60.$$

COROLLARY.

If the 3 first Terms of such an *Harmonic Proportion* be given, the 4th is easily found.

For if a, b, c, d , be *Harmon. Proportional*.

Then $a - b : c - d :: a : d$; and $a d - b d = a c - a d$

$$\text{Theref. } d = \frac{a c}{2 a - b}, \text{ and } a = \frac{b d}{2 d - c}.$$

DEFINITION III.

IF the Terms of an Harmonic Proportion be continued, then 'tis called an Harmonic Progression.

Thus, Supposing $\begin{cases} b \text{ to be the } 2^{\text{d}}. \text{ Term} \\ d \text{ the Difference of the } 1^{\text{st}}. \text{ and } 2^{\text{d}}. \end{cases}$

And that the $1^{\text{st}}.$ exceeds the $2^{\text{d}}.$

The Progression will be

$$b + d, b, \frac{b^2 + b d}{b + 2 d}, \frac{b^2 + b d}{b + 3 d}, \frac{b^2 + b d}{b + 4 d}, \frac{b^2 + b d}{b + 5 d}, \&c.$$

COROL

C O R O L L A R Y 3.

Whence, If out of a Rank of *Harmonic Proportionals* there be taken any *Series* of equidistant Terms, that *Series* will be *Harmonically Proportional*.

S C H O L I U M.

'Tis obſerved alſo, that *Harmonic Proportion* has ſeveral other *Properties* common with thoſe of *Ariſthmetic* and *Geometric Proportions*.

D E F I N I T I O N IV.

WHent three Terms are ſo diſpoſed, that the Diff. of the 1ſt. and 2d. : Diff. of the 2d. and 3d. :: 3d. : 1ſt. they are ſaid to be in a *Contra-Harmonic Proportion*.

Thus, 6, 5, 3, and 12, 10, 4, are *Contra-Harmonics*.
For $6 - 5 : 5 - 3 :: 3 : 6$; and $12 - 10 : 10 - 4 :: 4 : 12$

Or ſuppoſing b greater than n .

If the 2d. Term be greater than the 1ſt.:

Then $bn + n^2$, $b^2 + n^2$, $b^2 + bn$, are *Contra-Harmon.*

For $bn - b^2 : n^2 - bn :: b^2 + bn : bn + n^2$.

But if the 1ſt. Term exceeds the 2d.:

Then $b^2 + bn$, $b^2 + n^2$, $bn + n^2$, are *Contra-Harmon.*

For $bn - n^2 : b^2 - bn :: bn + n^2 : b^2 + bn$.

M

CHAP.

C H A P. V.

Of Fractions or Parts of Integers.

DEFINITION I.

ANY Whole Thing or an Unit, may be considered as Divided into any Number of Equal Parts, which have their Name from the Number contain'd in that Unit.

As if an Unit be conceiv'd to be divided into four equal Parts, those Parts are call'd *Fourths*, and are thus written, $\frac{1}{4}$.

And any Number of those Parts, suppose three are thus express'd, *Three Fourths*, or *Three divided by Four*, and thus written $\frac{3}{4}$, and called a FRACTION.

S C H O L I U M I.

Hence, the Number $\left\{ \begin{array}{l} \text{above} \\ \text{below} \end{array} \right\}$ the Line is $\left\{ \begin{array}{l} \text{Numerator} \\ \text{Denominator} \end{array} \right\}$ call'd the $\left\{ \begin{array}{l} \text{Numerator} \\ \text{Denominator} \end{array} \right\}$ expressing $\left\{ \begin{array}{l} \text{some of} \\ \text{all} \end{array} \right\}$ those Parts into which the Unit is Divided.

S C H O L I U M 2.

If the Numerator be $\left\{ \begin{array}{l} \text{greater than} \\ \text{equal to} \\ \text{less than} \end{array} \right\}$ the Denominator, the Fraction is $\left\{ \begin{array}{l} \text{greater than} \\ \text{equal to} \\ \text{less than} \end{array} \right\}$ the Unit, and is call'd $\left\{ \begin{array}{l} \text{an Improper} \\ \text{a Proper} \end{array} \right\}$ Fraction.

C O R O L.

C O R O L L A R Y 1.

Whence it follows, That a *Fraction* is but a *Quotient*, signifying a *Part* or *Parts* of an *Unit*, express'd by a *Numerator* as a *Dividend* and a *Denominator* as a *Divisor*.

Thus, *one Third* part of any *Thing* is the *Quotient* of that *Thing* *Divided* by *Three*, which by *Common Division* is expressed thus, $\frac{1}{3}$.

Also $\frac{3}{4}$ signifies *Three Fourth* parts of an *Unit*, or *One Fourth* part of *Three Units*.

For $\left\{ \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \right\}$ fourth of $\left\{ \begin{smallmatrix} 3 \text{ Units} \\ 1 \text{ Unit} \end{smallmatrix} \right\}$ is *Thrice one fourth* of *one Unit*.

C O R O L L A R Y 2.

As the *Numerator* is to the *Denominator* ;
So is the *Fraction* to *Unit*.

For the *Dividend* is to the *Divisor*,
As the *Quotient* is to *Unit*. by *Def. of Division*.

C O R O L L A R Y 3.

Fractions having the same *Denominators* are one to another as their *Numerators*.

Thus, $\frac{2}{3} : \frac{3}{3} :: 2 : 3$.

For $2 : \frac{2}{3} :: 5 : \frac{5}{3}$ } by preced. Cor.

And $3 : \frac{3}{3} :: 5 : \frac{5}{3}$ }

Theref. $\frac{2}{3} : \frac{3}{3} :: 2 : 3$, by Cor. 16. Th. 2. Ch. 3

COROLLARY 4.

As any *Fraction* is to *Unit*,
So is *Unit* to the *Reverse Fraction*.

Thus, $\frac{3}{4} : 1 :: 1 : \frac{4}{3}$.

For $\frac{3}{4} : 1 :: 3 : 4$ } by Cor. 2.

And $1 : \frac{4}{3} :: 3 : 4$ }

Theref. $\frac{3}{4} : 1 :: 1 : \frac{4}{3}$ by Cor. 16. Th. 2. Ch. 3.

LEMMA 1.

To Multiply the *Numerator* is to Multiply the *Fraction*; thus, $\frac{2 \times 3}{5} = \frac{2}{5} \times 3$.

For $5 : 2 :: 3 : \frac{2 \times 3}{5}$ by Case 1. of Conv. to Th. 2.

And $5 : 2 :: 1 : \frac{2}{5}$, by Def. of Divis.

Theref. $1 : \frac{2}{5} :: 3 : \frac{2 \times 3}{5}$ by Cor. 16. Th. 2. Ch. 3.

But $1 : \frac{2}{5} :: 3 : \frac{2}{5} \times 3$, by Def. of Mult.

Theref. $\frac{2 \times 3}{5} = \frac{2}{5} \times 3$.

LEMMA 2.

The *Terms* of a *Fraction* being *Multiplied* or *Divided*
by the same *Quantity* alter not its value.

Thus $\frac{2 \times 3}{2 \times 4} = \frac{6}{8}$

For $2 \times 3 (= 6) : 2 \times 4 (= 8) :: 3 : 4$, by Cor. 5. Th. 2.

Theref. $\frac{6}{8} = \frac{3}{4}$, by Def. of Proportion.

Alfo

Alſo, $\frac{5}{5} \frac{10}{15} (\frac{2}{3})$

For, $\frac{10}{5} : \frac{15}{5} :: 10 : 15$, by *Cor. 3. Def. 1.*

But, $\frac{10}{5} : \frac{15}{5} :: 2 : 3$.

Theref. $\frac{10}{5} = \frac{2}{3}$, by *Def. of Proportion.*

DEFINITION II.

TIS not only an Unit that may be Divided into any Number of Equal Parts, but alſo any of thoſe Parts may be divided infinitely into others call'd Compound Fractions, and thoſe again ſubdivided infinitely.

Theſe things being thoroughly underſtood, all Operations relating to *Fractions* admit of very few or no Difficulties.

DEFINITION III.

THE Changing of Quantities out of one Form or Denomination into another, (either for the more eaſe in Working, or Eſtimating of their Value) is called Reduction.

SCHOLIO-

S C H O L I U M.

Since it often happens in *Reducing, Adding, Subtracting, &c. Fractions*, that they swell into too great Numbers, which are not so managable as smaller ones ; therefore we shall in the next place shew the way of *Reducing* them into their *Least Terms*, either before or after such Operations, as there is occasion ; which is done by the help of the following

P R O B L E M.

Two Numbers being given, to find their Greatest Common Measure, (i. e.) the Greatest Number that can Divide both without Remainder.

R U L E.

Divide the Greatest by the Least, and that Divisor by the Remainder continually, till nothing remain, and the last Divisor will be the greatest Common Measure.

E X A M P L E.

If the Numbers be 152, and 184 :
Their Greatest Common Measure is 8.

$$\begin{array}{r}
 152 \overline{) 384} (2 \\
 \underline{304} \\
 80 \overline{) 152} (1 \\
 \underline{80} \\
 72 \overline{) 80} (1 \\
 \underline{72} \\
 8 \overline{) 72} (9 \\
 \underline{72} \\
 00
 \end{array}$$

For

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For, Suppose a and b were the Quantities, whose *Greatest Common Measure* is required.

Now suppose $a = 4b + c$.

And $b = 2c + d$.

Also $c = 3d$.

Then $b = 7d$, and $a = 31d$, therefore d is the *Greatest Common Measure* of a and b .

NOTE.

If 1 be the *Greatest Common Measure*, the Numbers are said to be *Prime* to one another.

PROPOSITION 1.

To reduce a Fraction into its Least Terms.

RULE.

Divide the Numerator and Denominator by their *Greatest Common Measure*, their Quotients will be a Fraction equivalent to the former, and in the Least Terms.

Numeral EXAMPLE.

If the given Fraction be $\frac{75}{135}$.

The *Greatest Common Measure* is 15.

And $\frac{15}{15} \frac{75}{135} (\frac{5}{9})$ the given Fraction in its least Terms.

For $\frac{5}{9} = \frac{75}{135}$ by Lem. 2.

And 15 is the *Greatest Number* that will Divide 75 and 135 without Remainder, by the foregoing Probl.

Theref. $\frac{5}{9}$ is the least Term $\frac{75}{135}$ can be brought to.

Note,

NOTE.

When the *Greatest Common Measure* is 1, the *Fraction* is already in the *Smallest Terms*.

For dividing by 1 do's not diminish them.

COROLLARY 1.

A *Fraction* whose *Terms* are *even*, may be abbreviated by a *Continual Division* by 2.

$$\text{Thus, } \frac{256}{384} = \frac{2}{3}$$

$$\text{For } \begin{array}{l} 2) \, 256 \mid 128 \mid 64 \mid 32 \mid 16 \mid 8 \mid 4 \mid 2 \\ 2) \, 384 \mid 192 \mid 96 \mid 48 \mid 24 \mid 12 \mid 6 \mid 3 \end{array}$$

COROLLARY 2.

So also may any *Terms*, that are found to be *Divisible* by any other *Digit*.

$$\text{Thus, } \frac{162}{1296} = \frac{2}{16} = \frac{1}{8}$$

$$\text{For } \begin{array}{l} 3) \, 162 \mid 54 \mid 18 \mid 6 \mid 2 \\ 3) \, 1296 \mid 432 \mid 144 \mid 48 \mid 16 \end{array}$$

COROLLARY 3.

When both *Terms* have *Cyphers* adjoining, cut off equal *Cyphers* from both.

$$\text{Thus, } \frac{1000}{3000} = \frac{1}{3}, \text{ for } \frac{1 \, 000}{3 \, 000} = \frac{1}{3}$$

$$\text{And } \frac{1000}{1400} = \frac{5}{7}, \text{ for } \frac{10 \, 00}{14 \, 00} = \frac{10}{14} = \frac{5}{7}$$

These *Corollaries* are evident from *Lemma 2*.

Literal

LITERAL EXAMPLES.

$$\frac{xn}{xy} = \frac{n}{y}; \frac{25x^2}{5x^2 + 15a^2} = \frac{5}{x+3a}; \frac{n^2xr^2}{xy^2r} = \frac{n^2r}{yy}.$$

This is evident from *Cor. to Lem. 2. Sect. 3.*

Also $\frac{nnn - naa}{nn + 2na + aa} = \frac{nn - na}{n + a}; \frac{x^2 - y^2}{x^2 + 2xy + y^2} = \frac{x - y}{x + y}.$

And $\frac{x^6 a^2 y^2 n^2 + 4x^6 a^2 n^3 m}{y^2 m^2 r^4 + 4nm^3 r^4} = \frac{x^6 a^2 n^2}{m^2 r^4}.$

PROPOSITION 2.

To Reduce an Integer into an Improper Fraction.

CASE I.

When there is no Denominator assign'd.

RULE.

Let the given Integer be a Numerator, and Unit its Denominator.

Thus, $2 = \frac{2}{1}; 15 = \frac{15}{1}; 500 = \frac{500}{1}.$

And $n = \frac{n}{1}; a + e = \frac{a + e}{1}.$

For Dividing by Unit does not diminish the Value.

N

CASE

CASE II.

Where there is a Denominator assign'd.

RULE.

Multiply the Integer by the assign'd Denominator, the Product shall be the Numerator.

Numeral EXAMPLE.

If $\left\{ \begin{matrix} 14 \\ 9 \end{matrix} \right\}$ be the given $\left\{ \begin{matrix} \text{Integer} \\ \text{Denominator} \end{matrix} \right\}$.

$$\text{Then } 14 \left(= \frac{9 \times 14}{9} \right) = \frac{126}{9}.$$

For $126 : 9 :: 14 : 1$. by Def. of Divis.

And $126 : 9 :: \frac{126}{9} : 1$ by Cor. 2. Def. 1.

Therefore $14 = \frac{126}{9}$.

$$\text{Or } 14 = \frac{14}{1} = \frac{9 \times 14}{9 \times 1} = \frac{9 \times 14}{9} = \frac{126}{9} \text{ by Lem. 2.}$$

Literal EXAMPLES.

$$x = \frac{nx}{n}; x + y = \frac{xn + yn}{n}; a = \frac{am + an}{m + n}.$$

For $x \times n = nx$; $x + y \times n = xn + yn$, &c.

PROPOSITION 3.

To Reduce mixt Fractions into Improper ones.

RULE.

R U L E.

Multiply the Integer by the Denominator of the Fraction, and add the Numerator to the Product, subscribing the same Denominator.

Numeral *EXAMPLES.*

$$36\frac{3}{4} (= \frac{36 \times 4 + 3}{4} = \frac{144 + 3}{4}) = \frac{147}{4}.$$

For the Unit here is consider'd as divided into 4 Equal Parts.

Therefore the Integers must be Multiplied by 4, to produce 4ths.

To which 3 fourths being Added, the Sum will be also 4ths.

Literal *EXAMPLES.*

$$1. \quad 2 + \frac{x n}{a} = \frac{2a + x n}{a}.$$

$$2. \quad a + n - \frac{x x}{a r} = \frac{a a r + a r n - x x}{a r}.$$

PROPOSITION 4.

To Reduce an Improper Fraction into an Integer, or Mixt Fraction.

R U L E.

Divide the Numerator by the Denominator, and the Quotient will be the Integer, or mixt Fraction required.

Thus, $\frac{126}{9} = 14$; and $\frac{147}{4} = 36\frac{3}{4}$.

For $\frac{126}{9} : 1 :: 126 : 9$, by *Cor. 2. Def. 1.*

And $14 : 1 :: 126 : 9$, by *Def. of Divis.*

Theref. $\frac{126}{9} = 14$.

Also, $\frac{ra + xn}{a} = r + \frac{xn}{a}$; and $\frac{am + an}{m + n} = a$.

PROPOSITION 5.

To Reduce a Fraction into its Equivalent that shall have any assign'd Denominator.

R U L E.

Multiply the Numerator of the Fraction by the assign'd Denominator, and Divide the Product by the Denominator of the Fraction; the Quotient shall be the Numerator required.

E X A M P L E.

To reduce $\frac{3}{4}$ into a Fraction, whose Denominator is 12.

$$\frac{3}{4} = \frac{4) 3 \times 12}{12} = \frac{4) 36}{12} = \frac{9}{12}.$$

$$\text{For } 4 : 3 :: 12 : \frac{3 \times 12}{4} = \frac{36}{4} = 9.$$

$$\text{Theref. } \frac{3}{4} (= \frac{4) 3 \times 12}{12} = \frac{9}{12} \text{ by Def. of Prop.}$$

COROL.

C O R O L L A R Y.

By this *Proposition*, *Fractions* are reduced into their *Known Parts of Time, Measure, Weight, Coin, &c.* As also into *Decimals, Sexageffimals, &c.* and the contrary.

$$\text{Thus, } \frac{3}{4} \text{ L.} = \left(\frac{3 \times 20}{4} \right) = 15 \text{ s.}$$

$$\text{And } \frac{2}{3} \text{ Deg.} = \left(\frac{2 \times 60}{3} \right) = 40'$$

$$\text{Also } \frac{7}{11} \text{ Deg.} = 38', 10'', 54''', 32'''' \text{, \&c.}$$

$$\text{For } 1. \frac{7}{11} \text{ Min.} = \left(\frac{7 \times 60}{11} \right) = 38' \frac{2}{11} \text{.}$$

$$2. \frac{2}{11} \text{ Min.} = \left(\frac{2 \times 60}{11} \right) = 10' \frac{10}{11} \text{.}$$

$$3. \frac{10}{11} \text{ Sec.} = \left(\frac{10 \times 60}{11} \right) = 54''' \frac{6}{11} \text{.}$$

$$4. \frac{6}{11} \text{ Thirds} = \left(\frac{6 \times 60}{11} \right) = 32'''' \frac{2}{11} \text{.}$$

&c.

S C H O L I U M.

Hence also, To reduce a *Fraction* into its *Equivalents*, that shall have any *assign'd Numerator*.

E X A M P L E.

To reduce $\frac{3}{4}$ into a *Fraction*, whose *Numerator* is 9.

$$\text{Then } \frac{3}{4} = \left(\frac{9}{3} \right) \frac{3}{4 \times 3} = \frac{9}{12} \text{.}$$

$$\text{For } 3 : 4 :: 9 : \frac{4 \times 9}{3} = 12 \text{.}$$

P R O.

PROPOSITION 6.

To reduce Fractions of Different Denominators, into their Equivalents, which shall have the same Denominator.

R U L E.

Multiply all the Denominators continually, for a Common Denominator, and each Numerator continually by the other's Denominators, for new Numerators.

Numerical EXAMPLES.

1. $\frac{2}{3}$ and $\frac{1}{4}$ make $\frac{8}{12}$ and $\frac{3}{12}$.

$$\left. \begin{array}{l} \text{For } \frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \\ \text{And } \frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12} \end{array} \right\} \text{by Lem. 2.}$$

2. $\frac{1}{2}, \frac{3}{4}, \frac{5}{7}$, make $\frac{21}{28}, \frac{15}{28}, \frac{20}{28}$, or $\frac{14}{28}, \frac{21}{28}, \frac{20}{28}$.

$$\text{For } \left\{ \begin{array}{l} \frac{1}{2} = \frac{1 \times 4 \times 7}{2 \times 4 \times 7} = \frac{28}{56} = \frac{14}{28} \\ \frac{3}{4} = \frac{3 \times 2 \times 7}{4 \times 2 \times 7} = \frac{42}{56} = \frac{21}{28} \\ \frac{5}{7} = \frac{5 \times 2 \times 4}{7 \times 2 \times 4} = \frac{40}{56} = \frac{20}{28} \end{array} \right\} \text{by Lem. 2.}$$

Literal EXAMPLES.

1. $\frac{a}{c}, \frac{c}{n}$ make $\frac{an}{cn}, \frac{cc}{cn}$.

2. $\frac{a}{x},$

2. $\frac{a}{x}, \frac{c}{m}, \frac{r n}{q}$, make $\frac{a m q}{x m q}, \frac{c x q}{m x q}, \frac{r n x m}{q x m}$.

For $\left\{ \begin{array}{l} \frac{a \times m \times q}{x \times m \times q} = \frac{a m q}{x m q} (= \frac{a}{x}) \\ \frac{c \times x \times q}{m \times x \times q} = \frac{c x q}{m x q} (= \frac{c}{m}) \\ \frac{r n \times x \times m}{q \times x \times m} = \frac{r n x m}{q x m} (= \frac{r n}{q}) \end{array} \right\}$ by Lem. 2.

Therefore they are Equivalent Fractions, by Prop. 1.

SCHOLIUM.

Hence, to find *Two Integers*, that shall be one to the other as two given *Fractions*.

Suppose the *Fractions* were $\frac{2}{3}$ and $\frac{1}{2}$.

Then $\frac{2}{3} : \frac{1}{2} :: \frac{2 \cdot 2}{3 \cdot 1} : \frac{1 \cdot 1}{2 \cdot 1}$ by this Prop.

But $\frac{2 \cdot 2}{3 \cdot 1} : \frac{1 \cdot 1}{2 \cdot 1} :: 14 : 15$ by Cor. 3. Def. 1.

Theref. $\frac{2}{3} : \frac{1}{2} :: 14 : 15$ by Cor. 16. Th. 2. Ch. 3.

PROPOSITION 7.

Addition and Subtraction of Common Fractions.

Note, If the *Numerators* are not equal, they must be reduc'd to such as have equal ones; then by this

RULE.

The Sum, or Difference of the Numerators set over the Common Denominator, shall be the Sum or Difference of the given Fractions.

Numeral

Numeral EXAMPLE.

Let the *Fractions* be $\frac{5}{9}$ and $\frac{2}{9}$.

$$\text{Then } \frac{5}{9} \pm \frac{2}{9} = \frac{5 \pm 2}{9} = \frac{7}{9} \text{ or } \frac{3}{9}.$$

For $\frac{5}{9} : \frac{2}{9} :: 5 : 2$, by *Cor. 3. Def. 1.*

And $\frac{5}{9} \pm \frac{2}{9} : \frac{2}{9} :: 5 \pm 2 : 2$. by *Compound. or Divid.*

But $\frac{5 \pm 2}{9} : \frac{2}{9} :: 5 \pm 2 : 2$. by *Cor. 3. Def. 1.*

Theref. $\frac{5}{9} \pm \frac{2}{9} : \frac{2}{9} :: \frac{5 \pm 2}{9} : \frac{2}{9}$ by *Cor. 16. Theor. 2.*

Conseq. $\frac{5}{9} \pm \frac{2}{9} = \frac{5 \pm 2}{9}$.

Literal Examples in *Addition.*

$$1. \frac{aa}{c} + \frac{cc}{c} = \frac{aa + cc}{c}.$$

$$2. \frac{aa + ax}{a - x} + a = \frac{2aa}{a - x}.$$

$$3. \frac{aa}{n} + \frac{cc}{m} + \frac{cc}{x} = \frac{a^2 mx + c^2 nx + c^2 nm}{nm x}.$$

$$4. \frac{125}{x^3 - 25x} + \frac{x - 25}{x^2 + 10x + 25} = \frac{x^3 + 30x^2 + 250x + 625}{x^4 + 5x^3 - 25x^2 - 125x}.$$

Literal Examples in *Subtraction.*

$$1. \frac{aa}{c} - \frac{cc}{c} = \frac{aa - cc}{c}.$$

$$2. \frac{n^4 + a^3 m}{n c m} - \frac{a^3}{n c} = \frac{n n n}{c m}.$$

3. $2 a n$

$$3. \frac{2an + n^2}{a + n} - n = \frac{an}{a + n}$$

$$4. \frac{a^3 + n^3}{cx - x^2} - \frac{nnnn}{aac - aax} = \frac{a^3 + a^2n^3 - n^4x}{a^2cx - a^2x^2}$$

The Reason is the same with that in Numbers.

PROPOSITION 8.

Multiplication of Common Fractions.

RULE.

Multiply the { Numerators } for a new { Numerator.
Denominators } Denominator.

Numeral EXAMPLE.

$$\frac{3}{4} \text{ by } \frac{2}{3} \text{ make } \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2} = \frac{2}{4}$$

$$\text{For } \frac{4 \times 2}{4 \times 3} = \frac{8}{12} = \frac{2}{3}, \text{ by Lem. 2.}$$

And $\frac{6}{12} : \frac{2}{4} :: 6 : 8$ by Cor. 1. Def. 1.

That is, $\frac{6}{12} : \frac{2}{4} :: 3 : 4$

Or, $\frac{6}{12} : \frac{2}{4} :: \frac{3}{4} : 1$

Theref. $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$, by the Nature of Proportion

NOTE.

If { Mixt } Fractions, are to be Mul- { Improper } ones
: { Compound } tiplied, reduce them to { Single }

2. If one of the Factors be a whole Number, it must be made an Improper Fraction.

Literal EXAMPLES.

$$1. \frac{a}{n} \times \frac{c}{m} = \frac{ac}{nm}$$

For



For if $x = \frac{a}{n}$, and $r = \frac{c}{m}$, by *Substit.*

Then $a = xn$, and $c = rm$, by *Mult.*

$$\text{And } \left(\frac{xnrm}{nm} \right) = \frac{ac}{nm} (=xr) = \frac{a}{n} \times \frac{c}{m}$$

$$2. \frac{a^2 + 2ab + b^2}{ab} \times \frac{bb}{a+b} = \frac{ab + b^2}{a}$$

$$3. \frac{an}{m} \times a = \frac{aan}{m}; \text{ and } \frac{aaa}{an + nn} \times a + n = \frac{a^3}{n}$$

$$4. a + \frac{cc}{a-c} \times a - c = aa - ac + cc$$

$$5. a + \frac{nn}{a-n} \times a - 2n + \frac{nn}{a} = a^2 - 2an + 2n^2 - \frac{n^3}{a}$$

SCHOLIUM 1.

The Product of any Quantity Multiplied by a *Proper Fraction*, is always *Less* than that Quantity.

For in *Multiplying* by an *Unit*, the *Product* will be equal to the *Multiplicand*.

But a *Less Multiplier* gives a *Less Product* :

Therefore *Multiplying* by a *Proper Fraction*, (i. e. by *Less* than *Unit*) the *Product* must be *Less* than the *Multiplicand*.

SCHOLIUM 2.

Whence the *Product* of two *Proper Fractions* must be *Less* than either of them.

Thus,

Thus, $\frac{2}{3} \times \frac{4}{7} = \frac{8}{21}$, and $\frac{8}{21}$ is Less than either $\frac{2}{3}$ or $\frac{4}{7}$.

For $1 : \frac{4}{7} :: \frac{2}{3} : \frac{8}{21}$, by *Def. of Mult.*

But $\left\{ \begin{array}{l} 1 \\ \text{And} \end{array} \right\} > \left\{ \begin{array}{l} \frac{4}{7} \\ \frac{2}{3} \end{array} \right\}$ theref. $\left\{ \begin{array}{l} \frac{2}{3} \\ \frac{4}{7} \end{array} \right\} > \frac{8}{21}$.

PROPOSITION 9.

Division of Common Fractions.

R U L E.

Multiply the Dividend by the Divisor's Reverse Fraction; Or, (which is the same) Imagine the Terms of the Divisor changed, then work as in Multiplication.

Numeral EXAMPLE.

Thus $\frac{2}{3} \div \frac{4}{7} (= \frac{14}{12} = \frac{7}{6} = 1 \frac{1}{6})$.

That is, $\frac{2}{3} \times \frac{7}{4} = \frac{14}{12}$.

For $1 : \frac{4}{7} :: \frac{2}{3} : \frac{14}{21}$, by *Def. of Mult.*

But $\frac{2}{3} : 1 :: 1 : \frac{3}{2}$, by *Cor. 4. Def. 1.*

And $\frac{4}{7} : 1 :: \frac{7}{4} : \frac{7}{4}$, by *Cor. 15. Tb. 2. Ch. 3. §. 3.*

Theref. $\frac{4}{7} \div \frac{2}{3} = \frac{14}{12}$, by the *Nature of Proporti.*

N O T E.

1. If either *Dividend* or *Divisor* be *whole* or *mixt* Numbers, or if both be *Mixt* Numbers; *Reduce them to improper Fractions.*

2. If they are *Compound Fractions*; *Reduce them to single ones.*

Literal EXAMPLE.

$$1. \frac{a}{c} \div \frac{x}{n} = \frac{na}{xc}; \text{ and } \frac{ar}{x\bar{r}} \div \frac{m}{n} = \frac{nar}{mx\bar{r}}$$

$$2. \frac{n+3}{5} \div \frac{20a}{3} = \frac{3n+9}{100a}; \text{ and } \frac{aa}{n} \div aa = \frac{1}{n}$$

$$3. \frac{xxx}{nn} \div a = \frac{xxx}{ann}; \text{ and } a \div \frac{xxx}{nn} = \frac{ann}{xxx}$$

$$4. a-c \div \frac{a-c}{n} = n; \text{ and } \frac{nnxx}{r} \div \frac{nx}{r} = nx$$

$$5. \frac{a^2+2an+n^2-ar-nr}{an+ar-nr-r\bar{r}} \div \frac{a+n-r}{n+r} = \frac{a+n}{a-r}$$

Which may be *Demonstrated* thus.

For $\frac{x}{n}, \frac{a}{c} = \frac{xc}{nc}, \frac{an}{nc}$, by *Prop. 6.*

But $\frac{xc}{nc} : \frac{an}{nc} :: xc : an$, by *Cor. 3. Def. 1.*

i.e. $\frac{x}{n} : \frac{a}{c} :: xc : an$, by *Equal Division.*

Theref. $\frac{a}{c} \div \frac{x}{n} = \frac{an}{xc}$. by *Def. of Proport.*

S C H O L I U M 1.

If the *Fractions* are of one *Denomination*,

Then cast off that *Denominator*, and Divide the *Numerators*.

Because *Fractions* having the same *Denominators*, are as their *Numerators*, by *Cor. 3. Def. 1.*

S C H O.

SCHOLIUM 2.

The Quotient of any Quantity divided by a Proper Fraction is always Greater than that Quantity.

For in Dividing by Unit, the Quotient will be equal to the Dividend ;

But a Less Divisor gives a Greater Quotient :

Therefore in Dividing by a Proper Fraction, (i. e. by Less than an Unit) the Quotient must be Greater than the Dividend.

Or thus, $\frac{2}{3} \div \frac{4}{7} (= \frac{6}{7})$, And $\frac{6}{7}$ is Greater than $\frac{4}{7}$.

For $1 : \frac{6}{7} :: \frac{2}{3} : \frac{4}{7}$, by Def. of Divis.

And $1 : \frac{2}{3} :: \frac{6}{7} : \frac{4}{7}$, by Alternation.

But $1 > \frac{2}{3}$, theref. $\frac{6}{7} > \frac{4}{7}$.

PROPOSITION 10.

To reduce Compound Fractions into Single ones.

RULE.

Multiply $\left\{ \begin{array}{l} \text{Numerators} \\ \text{the } \end{array} \right\}$ continually $\left\{ \begin{array}{l} \text{Numerator.} \\ \text{for a new } \end{array} \right\}$ Denominator.

EXAMPLE.

Thus, $\frac{2}{3}$ of $\frac{4}{7}$ being Reduced, is $\frac{2 \times 4}{3 \times 7} = \frac{8}{21}$

For if $\frac{4}{7} = n$, then $\frac{2}{3}$ of $\frac{4}{7}$ is $\frac{2}{3} n = \frac{2}{3} \times n$

Therefore $\frac{2}{3}$ of $\frac{4}{7} = \frac{2 \times 4}{3 \times 7} = \frac{8}{21}$

DEFINI-

DEFINITION IV.

AN Unit may be imagin'd to be equally divided into 10 Parts, and each of those into 10 more; so that by a continual Decimal Subdivision, the Unit may be supposed to be Divided into 10, 100, 1000, &c. equal Parts, call'd, 10th, 100th, 1000th, Parts of an Unit.

And since Integers Increase from Unit, towards the Left-hand, in a Decuple Proportion, so that a Figure in any Place is Ten times as much as the same in the next Place below it, and but a Tenth part of what it signifies in the next Place above it; therefore as the 1st. 2d. 3d. &c. Place above that of Units, is Tens, Hundreds, Thousands, &c. So the 1st. 2d. 3d. &c. Place below that of Units, is Tenths, Hundredths, Thousandths, &c. Decreasing in a Subdecuple Proportion, as is evident from the following Table.

Integers.					Parts.				
Bill.	Millions		Units		Unitth		Millionth		Bill.
un.	th.	un.	th.	un.	un.	th.	un.	th.	un.
x	x	x	x	x	x	x	x	x	x
5	6	7	8	9	9	8	7	6	5
5	6	7	8	9	9	8	7	6	5
5	6	7	8	9	9	8	7	6	5

Whence

Whence those *Fractions*, whose *Denominators* are an *Unit* with a *Cypher* or *Cyphers*, are called *Decimal Fractions*, and may be written without their *Denominators*, distinguished by a *Point* or *Comma* prefix'd, and read like *Integers*, giving them the name of the last *Place* to the *Right-hand*.

Thus, $\left\{ \begin{array}{l} .5 \\ .25 \end{array} \right\}$ signifies $\left\{ \begin{array}{l} 5 \text{ Tenths.} \\ 25 \text{ Hundredths.} \end{array} \right.$

C O R O L L A R Y.

As $\left\{ \begin{array}{l} \text{Cyphers before} \\ \text{Integers} \\ \text{Decimals} \end{array} \right\}$, (advancing them so many Places towards the $\left\{ \begin{array}{l} \text{Left} \\ \text{Right} \end{array} \right\}$ hand) $\left\{ \begin{array}{l} \text{Increase} \\ \text{Diminish} \end{array} \right\}$ their value.

S C H O L I U M.

1. Hence it is, That 1, 2, 3, &c. *Cyphers* before a *Decimal*, advance it so many Places forward, whereby 'tis made 10, 100, 1000, &c. times less. Thus,

$\left\{ \begin{array}{l} .25 \dots \\ .025 \dots \\ .0025 \dots \\ .00025 \dots \end{array} \right\}$ signifies $\left\{ \begin{array}{l} 25 \text{ Hundredths} \\ 25 \text{ Thousandths} \\ 25 \text{ Ten Thousandths} \\ 25 \text{ Hundred Thousandths.} \end{array} \right.$

2. Therefore, a Figure in the 1st. 2d. 3d. &c. *Decimal Place*, is 10, 100, 1000, &c. times less than if it were an *Integer*.

3. Consequently, each removal of a *Figure* into a *Place* forward makes it *Ten* times less than it was before.

P R O

PROPOSITION II.

To reduce Vulgar Fractions into Decimals.

R U L E.

To the Numerator add as many Cyphers as you would have Decimal Places; then divide it by the Denominator, and the Quotient (if there be no Remainder) will be a Decimal equivalent to the vulgar Fraction given.

But when there is a Remainder, you may by adding more Cyphers, proceed so as to bring the Quotient nearly equal.

$$\text{Thus, } \frac{1}{2} = \frac{1.0}{2} = .5, \text{ for } 2 : 1 :: 1.0 : .5$$

$$\frac{3}{4} = \frac{3.00}{4} = .75, \text{ for } 4 : 3 :: 1.00 : .75$$

$$\frac{2}{7} = \frac{2.000000 \&c.}{7} = .285714 = \frac{2}{7} \text{ near, not wanting } \frac{1}{1000000} \text{ part of an Unit.}$$

$$\text{For } 7 : 2 :: 1.000000, \&c. : .285714, \&c.$$

S C H O L I U M.

'Tis observed when a Fraction is reduced to the *smallest Terms* :

That if its *Denominator* be Compounded only of the *Prime Numbers* 2 and 5 (the *Components* of 10) the *Decimal* of that *Fraction* will be determin'd.

But if the *Denominator* be Compounded of any other *Prime Numbers*, it will be Indetermined : and the same Figures will return again in order, and continue to *Circulate*, either by one Figure, or by two, three, &c. Figures, tho' never by more than the Number of *Units* in the *Denominator* less 1.

For

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For the *Remainder* being always leſs than the *Diviſor*, therefore may be any *Number* leſs by 1 than it :

But in ſo many *Operations*, at moſt, as there are *Units* in the *Diviſor*, one of the *Remainders* muſt return again ;

Therefore, the ſame *Figure* in the *Quotient* muſt alſo return, and ſo continue the *Circulation*.

To find the Number of the Circulating Figures.

1. Divide the *Denominator* by 2 and 5 as often as poſſible ; if it come to be 9, 99, 999, 9999, &c. or an *Aliquot part* of ſuch *Number*, or a *Number Compounded* of 2 or 5, and ſuch *Aliquot Part* ; Then the *Number of the Circulating Figures*, will be equal to ſo many *Figures of 9*, as there are in the *Number found*.

2. If one of the *Prime Numbers* compounding the *Denominator* (excluding thoſe of 2 and 5) be not an *Aliquot part* of the other :

Then the *Number of the Circulating Figures* will be equal to the *Product of them required by thoſe Compounding Prime Numbers*.

3. And when the *Denominator* is *Compounded* of 2, or 5, or any *Power* of them ; Then the *Circulating Figures* begin at ſuch a *Place of Decimals*, as is denoted by the *Index of 2 or 5*, *ſuppoſed* in that *Compoſition*, more 1.

PROPOSITION 12.

Addition and Subduction of Decimal Fractions.

P

R U L E.

R U L E.

Place every Figure under that of the like Name, and Add, or Subduct, as if they were Integers.

Thus,

To 34.25	From 16.5
Add 3.026	Subd. .125
Sum 37.276	Rem. 16.375

The Reason is the same with that of Integers.

P R O P O S I T I O N 13.

Multiplication of Decimal Fractions.

R U L E.

Multiply the Factors as if all were Integers; And the Decimals in the Product, must be equal to the Sum of those in both Factors; if they are not, prefix Cyphers to supply the Defect.

For the Index of each Figure in the Product, must be equal to the Sum of the Indices of the Multiplied and Multiplying Figures.

Thus,

Mult. 3.52 by 4.3, the Product is 15.136

$$\text{For } \left\{ \begin{array}{l} 3.52 = \frac{352}{100} \\ 4.3 = \frac{43}{10} \end{array} \right\} \text{ and } \frac{352}{100} \times \frac{43}{10} = \frac{15136}{1000} = 15.136$$

Also, .013 Mult. by .005, gives .000065

$$\text{For } \left\{ \begin{array}{l} .013 = \frac{13}{1000} \\ .005 = \frac{5}{1000} \end{array} \right\} \text{ and } \frac{13}{1000} \times \frac{5}{1000} = \frac{65}{1000000} = .000065$$

SCHOLIO-

S C H O L I U M 1.

When a *Decimal*, or *Mix'd Number* is to be *Multiplied* by an *Unit* with *Cyphers*; 'Tis but removing the *Point* or *Comma* ſo many *Places* towards the *Right-hand* in the *Multiplicand*, as there are *Cyphers annex'd* to the *Unit*.

Thus,

$$2537 \text{ Mult. by } \left\{ \begin{array}{l} 10 \\ 100 \\ 1000 \\ 10000 \\ 100000 \end{array} \right\} \text{ Prod. is } \left\{ \begin{array}{l} 2.537 \\ 25.37 \\ 253.7 \\ 2537 \\ 25370 \end{array} \right.$$

S C H O L I U M 2.

In *Multiplication*, if it were required to find only an *assign'd part* of the *Product*.

1. Set the *Unit Place* of the *Leſſer Number* under that *Place* of the *Greater*, whoſe *Index* is equal to that of the *deſign'd Right-hand Place* of the *Product* (i. e. the *first in Integers*, and the *laſt in Decimals*) or to the *Number of Figures* to be cut off in the *Integers*, or left in the *Decimals*.

2. Then ſet the reſt of the *Figures* of the *Leſſer Number* in a contrary *Order*.

3. And begin to *Multiply* always at that *Figure* of the *Multiplicand*, which ſtands over the *Multiplier*.

Note, That particular regard muſt be had of the *Increase* that would ariſe from the foregoing *Figures* of the *Multiplicand*.

In the Prod. of $\overset{210777}{798.0625}$

By $\overset{1877}{78.54}$
Required the *Integers* only.

$$\begin{array}{r} 798.0625 \\ 45.87 \\ \hline 35864 \\ 6384 \\ 399 \\ 31 \\ \hline \end{array}$$

62679 Required.

In the Prod. of $\overset{2107}{419.3}$

By $\overset{0777}{0.6375}$
Reqd. the *Integers* only

$$\begin{array}{r} 419.3 \\ 5736.0 \\ \hline 251 \\ 13 \\ 3 \\ \hline \end{array}$$

267 Required.

In the Prod. of $\overset{210777}{798.0625}$

By $\overset{1077}{78.54}$
Reqd. 2 *Decimal Places.*

$$\begin{array}{r} 798.0625 \\ 45.87 \\ \hline 3586437 \\ 638450 \\ 39903 \\ 3192 \\ \hline \end{array}$$

62679.82 Reqd.

In the Prod. of $\overset{210777}{535.5625}$

By $\overset{1777}{.6375}$
Reqd. 3 *Decimal Places*

$$\begin{array}{r} 535.5625 \\ 5736.0 \\ \hline 321338 \\ 16067 \\ 3749 \\ 267 \\ \hline \end{array}$$

341.421

The Reason of this *Contraction* is obvious:

For the *Index* of the *Right-hand Figure* of any *Product* is, the *Sum* of the *Indices* of the *Factors*;

And by *inverting* the position of the *Figures* (as the *Rule* directs) the *Sums* of the *Indices* of each Corresponding *Place* in the *Factors*, will be equal among themselves; and therefore equal to the *Index* of the *Right-hand-Place* of the *Product* required:

But

But *Products*, whose *Indices* are equal, belong to the same *Place*, therefore must be set under each other, and their *Sum* must be the *Product* required.

PROPOSITION 14.

Division of Decimal Fractions.

R U L E.

Divide as if all were Integers; (annexing Cyphers to the Dividend, if need be): And

Let the first Figure in the Quotient be of the same Name (i. e. have the same Index) with that Figure of the Dividend, which stands, (or is imagin'd to stand) over the Unit Place of the Divisor.

For the Index of each Figure in the Quotient, must be equal to the Index of the Divided Figure less by the Index of the Dividing Figure.

Or, the Decimal Places in the Divisor and Quotient must be equal to those in the Dividend; If they are not prefix Cyphers to the Quotient, to supply the Defect.

For the Dividend is equal to the Product of the Divisor and Quotient;

But both Factors contain as many Decimal Places as the Product does:

Therefore, what Decimal Places are in the Dividend more than in the Divisor, must be supplied in the Quotient.

Thus,

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Thus, .0325 Divided by .25, gives .13 in the Quotient.

$$\text{For } \left\{ \begin{array}{l} .0325 = \frac{325}{10000} \\ .25 = \frac{25}{100} \end{array} \right\} \text{ and } \frac{325}{10000} \div \frac{25}{100} = \frac{325}{10000} \times \frac{100}{25} = \frac{325}{250} = .13$$

S C H O L I U M.

1. If the Divisor be Greater, or have more Decimal Places, than the Dividend; then by annexing Cyphers to the Dividend, the Quotient may be had to any accuracy. Thus; .25) .07864, 00 (.31456.

2. Therefore, when there is a Remainder after Division, (tho' neither Dividend or Divisor consist of any Decimals) 'tis but adding Cyphers to the Dividend, and so proceed to any exactness.

3. When a Decimal or Mix'd Number is to be Divided by an Unit with Cyphers, 'Tis but removing the Pointer Comma in the Dividend, so many Places further towards the Left-hand, as there are Cyphers annex'd to the Unit, prefixing Cyphers to the Dividend to supply vacancy, if need be. Thus,

$$253.7 \text{ Divided by } \left\{ \begin{array}{l} 10 \\ 100 \\ 1000 \\ 10000 \end{array} \right\} \text{ Quotient is, } \left\{ \begin{array}{l} 25.37 \\ 2.537 \\ .2537 \\ .02537 \end{array} \right.$$

4. In Dividing by an Infinite Number, the Division may oftentimes be very usefully Contracted, by

Taking as many of the Left-hand Figures of the Divisor as you think convenient, for the first Divisor, by which Divide the given Number, and omit one Figure of the Divisor at each following Operation.

Thus,

Thus,

$$\begin{array}{r|l}
 10986, 122 | 86345 & 47712127 \\
 \dots & (4342, 94 - \\
 & 3767639 \\
 & 471803 \\
 & 32359 \\
 & 10387 \\
 & 505 \\
 & 69 \\
 & 5c.
 \end{array}$$

PROPOSITION 15.

To reduce a Decimal into a Common Fraction.

R U L E.

Multiply the Numerator of the Decimal by the Denominator of the Common Fraction.

Thus, .575 of a *l.* is 11 *s.* 6 *d.*

.575 For 1000 : ,575 :: 20 *s.* : 11 *s.* $\frac{5}{10}$.

$$\begin{array}{r|l}
 & 20 \\
 \hline
 s. 11 & .500 \\
 & 12 \\
 \hline
 d. 6 & .000
 \end{array}
 \quad \text{And } 10 : ,5 :: 12d. : 6d.$$

PROPOSITION 16.

To find a Multiplier that shall Effect the same as a given Divisor.

C A S E I.

If the Divisor be an Integer; make it a Denominator, and 1 the Numerator, that Fraction shall be the Multiplier required.

For

$$\text{For } \frac{N}{D} = N \times \frac{1}{D}.$$

C A S E II.

If the *Divisor* be a *Fraction*.

Its Reverse Fraction shall be the Multiplier required.

$$\text{for } \frac{n}{d} \div \frac{N}{D} (= \frac{d}{n} \times \frac{N}{D}.$$

N O T E.

Hence, *Divisor* given : 1 :: 1 : *Multiplier* required.

And by finding of a *Factor*, when a *Divisor* is given; or a *Divisor*, when a *Factor* is given; one may advantageously Contract several tedious Operations in *Arithmetic*, and find variety of excellent useful Rules, for ease and expedition in *Common* and *Practical Accounts*.

C H A P. VI.

The Arithmetic of Incommensurables.

DEFINITION I.

Those Quantities, which have the same Common Measure, are said to be Commensurable.

S C H O L I U M I.

Therefore Commensurable Quantities may be expressed by Numbers; or are as Number to Number.

Q

DEFF

DEFINITION II.

AND those Quantities which have not the same Common Measure, are said to be Incommensurable.

SCHOLIUM 1.

Therefore *Incommensurable Quantities* are not as *Number to Number*, that is, their Magnitudes cannot be express'd by *Numbers*; yet they may be *Commensurable in Power*.

SCHOLIUM 2.

Therefore the *Roots of Imperfect Powers* are *Incommensurable Quantities*, and are usually call'd *Surd Roots*, because inexpressible by any known way of *Notation*, otherwise than by their *Radical Signs* or *Indices*.

And tho' *Incommensurable Quantities* have no *Ratio of Number to Number* with *Commensurable ones*, yet they may be *Commensurable one to another*.

PROPOSITION I.

To reduce Roots into their most Simple Expressions.

RULE.

Divide the given Power by the greatest Power, denoted by the Index, contained therein, that Measures it without Remainder; let the Quotient be affected by the Radical Sign, and have the Root of the Divisor prefix'd, or connected by the Sign $\times$.

Thus, to reduce $a^{\frac{1}{2}}$ into its most simple Expressions or Lowest Term.

Suppose

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Suppose x^n the greatest n Power, that will Divide a without Remainder.

Let $y = \frac{a}{x^n}$, then will $a^{\frac{1}{n}} = x \times y^{\frac{1}{n}}$

For $a = x^n y$, theref. $a^{\frac{1}{n}} (= \overline{x^n y})^{\frac{1}{n}} = \overline{x^n}^{\frac{1}{n}} \times y^{\frac{1}{n}} = x \times y^{\frac{1}{n}}$.

So $a^3 y - a^2 y^2 + 2a^2 y x + a y x^2 - a y^3 + y^2 x^2 - 2y^2 x + y^4$

will be $\overline{a + x - y \times a y + y^2}^{\frac{1}{2}}$.

And, $a^6 - 9a^5 + 27a^4 - 15a^3 - 108a^2 + 324a - 324$

will be $\overline{a - 3 \times a^3 + 12}^{\frac{1}{2}}$.

Also $\overline{75}^{\frac{1}{2}}$ and $\overline{27}^{\frac{1}{2}}$ will be $5 \times 3^{\frac{1}{2}}$ and $3 \times 3^{\frac{1}{2}}$

For $\frac{75}{25} = \frac{27}{9} = 3$, by the *Rule*

Theref. $\frac{\overline{75}^{\frac{1}{2}}}{5} = \frac{\overline{27}^{\frac{1}{2}}}{3} = 3^{\frac{1}{2}}$, by *Evolution*

Then, $\overline{75}^{\frac{1}{2}} = 5 \times 3^{\frac{1}{2}}$, and $\overline{27}^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}}$ by *Mult.*

PROPOSITION 2.

To reduce Roots of Different Names, into those of the same Name.

RULE.

Involve the Powers Reciprocally according to each others Index, for new Powers;

And let the Product of the Indices, be the Common Index.

Q 2

Thus,

Thus, $a^{\frac{1}{n}}$ and $e^{\frac{1}{m}}$ will be $\overline{a^m}^{\frac{1}{nm}}$ and $\overline{e^n}^{\frac{1}{nm}}$

$\overline{axx}^{\frac{1}{n}}$ and $\overline{ezyyy}^{\frac{1}{m}}$, will be $\overline{axx}^m \overline{ezyyy}^n \overline{e^{\frac{1}{nm}}}$ and $\overline{ezyyy}^{\frac{1}{n}} \overline{e^{\frac{1}{nm}}}$

Also $b \times a^{\frac{1}{n}}$ and $c \times e^{\frac{1}{m}}$, will be $b \times \overline{a^m}^{\frac{1}{nm}}$ and $c \times \overline{e^n}^{\frac{1}{nm}}$

This will be evident, when 'tis considered

That any Root, $\overline{xx}^{\frac{1}{2}} = \overline{x^2}^{\frac{1}{2}} = \overline{x^4}^{\frac{1}{4}} = \overline{x^8}^{\frac{1}{8}}$, &c.

COROLLARY 1.

Hence, *Rational Quantities* may be reduced to the *Form* of any assign'd Root.

Thus, a reduced to the *Form* of $x^{\frac{1}{n}}$ is $\overline{a^n}^{\frac{1}{n}}$

COROLLARY 2.

Also, *Roots with Rational Coefficients* may hence be reduced so as to be wholly affected by the *Radical Sign*.

Thus, $a \times e^{\frac{1}{n}} = \overline{a^n e}^{\frac{1}{n}}$, and $a y^2 \times \overline{x^3 r}^{\frac{1}{h}} = \overline{x^3 r a y y}^{\frac{1}{h}} \overline{e^{\frac{1}{n}}}$

COROLLARY 3.

Hence is known, whether any two given *Roots* are *Commensurable* one to another; As also, to find their *Ratio*.

For after *Reduction into the lowest Terms*, and the same *Name*;

If the *Powers* are equal, The *Roots* are *Commensurable*; And their *Ratio* is equal to that of the *Rational Coefficients*.

Thus,

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Thus, $\sqrt{75}$ and $\sqrt{27}$ reduc'd, will be $5\sqrt{3}$ and $3\sqrt{3}$ which are *Commenfurable*.

For $5\sqrt{3} : 3\sqrt{3} :: 5 : 3$, by *Cor. 5. Th. 2. Ch. 3.*

PROPOSITION 3.

Multiplication and Division of Simple Roots.

R U L E.

Let the Product, or Quotient, of the Powers be affected by the Radical Sign, prefixing the Product, or Quotient, of the Rational Coefficients (if there be any).

Examples in Multiplication.

Thus, $a^{\frac{1}{n}} \times e^{\frac{1}{n}} = \overline{ae}^{\frac{1}{n}}$; and $\overline{ar}^{\frac{1}{n}} \times \overline{ex}^{\frac{1}{n}} = \overline{aerx}^{\frac{1}{n}}$

Also, $a^{\frac{1}{n}} \times e = \overline{ae^n}^{\frac{1}{n}}$; and $a^{\frac{1}{n}} \times e^{\frac{1}{m}} = \overline{a^m e^n}^{\frac{1}{nm}}$

For $1 : a :: e : ae$, by *Def. of Multiplication*.

But $1^{\frac{1}{n}} : a^{\frac{1}{n}} :: e^{\frac{1}{n}} : \overline{ae}^{\frac{1}{n}}$, by *Cor. 4. Theor. 2. Ch. 3.*

Theref. $a^{\frac{1}{n}} \times e^{\frac{1}{n}} = \overline{ae}^{\frac{1}{n}}$, by *Conv. Theor. 2. Ch. 3.*

Examples in Division.

Thus, $a^{\frac{1}{n}} \div e^{\frac{1}{n}} = \frac{\overline{a}}{e}^{\frac{1}{n}}$, and $b a^{\frac{1}{n}} \div c a^{\frac{1}{n}} = \frac{b}{c}$

Also $a \div e^{\frac{1}{n}} = \frac{\overline{a^n}}{e}^{\frac{1}{n}}$, and $a^{\frac{1}{n}} \div e^{\frac{1}{m}} = \frac{\overline{a^m}}{e}^{\frac{1}{nm}}$

For

For $e : 1 :: a : \frac{a}{e}$ by Def. of Division.

But $e^{\frac{1}{n}} : 1^{\frac{1}{n}} :: a^{\frac{1}{n}} : \left| \frac{a}{e} \right|^{\frac{1}{n}}$ by Cor. 4. Th. 2. Ch. 3.

Theref. $a^{\frac{1}{n}} \div e^{\frac{1}{n}} = \left| \frac{a}{e} \right|^{\frac{1}{n}}$ by Conv. Th. 2, Ch. 3.

NOTE.

If the Powers be the same; Then the Power affected by the Sum, or Difference of the Indicas, shall be the Product, or Quotient sought.

In MULTIPLICATION.

1. $y^{\frac{n}{m}} \times y^{\frac{r}{s}} = y^{\frac{ns+mr}{ms}}$; and the $\frac{n}{m}$ Power of $y^{-\frac{r}{s}}$ is $y^{-\frac{nr}{ms}}$

2. $\frac{1}{y} \times \frac{1}{y^{\frac{1}{m}}}$, or $y^{-1} \times y^{-\frac{1}{m}} = y^{-\frac{m+1}{m}} = \frac{1}{y^{\frac{m+1}{m}}}$

3. $\frac{1}{y^{\frac{n}{m}}} \times \frac{1}{y^{\frac{r}{s}}}$, or $y^{-\frac{n}{m}} \times y^{-\frac{r}{s}}$ will be

$$y^{-\frac{sn+rm}{sm}} = \frac{1}{y^{\frac{sn+rm}{sm}}} = \frac{1}{y^{\frac{sn+rm}{sm}}}; \text{ Also}$$

4. $\frac{1}{a y^{\frac{1}{n}}} \times \frac{1}{y^{\frac{1}{s}}}$, or $a^{-\frac{1}{n}} \times y^{-\frac{1}{n}} \times y^{-\frac{r}{s}}$ will be

$$a^{-\frac{1}{n}} \times y^{-\frac{sm+rn}{sn}} = \frac{1}{a^{\frac{1}{n}} \times y^{\frac{sm+rn}{sn}}}, \text{ or}$$

$$\frac{1}{a^{\frac{1}{n}} \times y^{\frac{sm+rn}{sn}}}$$

In

IN DIVISION.

1. $y^{\frac{n}{m}} \div y^{\frac{r}{s}}$ will give $y^{\frac{ns-rm}{sm}}$

And $y^{-\frac{1}{n}} \div y^{\frac{1}{m}} = y^{-\frac{1}{n} - \frac{1}{m}} = y^{-\frac{m+n}{mn}}$

2. $\frac{1}{y^{\frac{1}{n}}} \div \frac{1}{y}$ or $y^{-\frac{1}{n}} \div y^{-1} = y^{-\frac{n-1}{n}} = \frac{1}{y^{\frac{n-1}{n}}}$

PROPOSITION 4.

Addition and Subtraction of like Simple Roots.

General RULE.

Let the given Roots be Connected by the Sign +, or —.

1. But if the Roots are Commensurable; then,

Since in any two Quantities, a and c , $\frac{a}{c} \pm 1 \times c = a \pm c$.

Therefore,

$$\left\{ \begin{array}{l} 8^{\frac{1}{2}} + 2^{\frac{1}{2}} = 10^{\frac{1}{2}} \\ 18^{\frac{1}{2}} - 2^{\frac{1}{2}} = 16^{\frac{1}{2}} \end{array} \right\} \text{ and } \left\{ \begin{array}{l} 768^{\frac{1}{4}} + 3^{\frac{1}{4}} = 1875^{\frac{1}{4}} \\ 1875^{\frac{1}{4}} - 3^{\frac{1}{4}} = 768^{\frac{1}{4}} \end{array} \right.$$

2. And if the Roots are reduced into their lowest Terms ;

Then the Sum, or Difference, of the Rational Coefficients Multiplied by the Common Root, will be the Sum, or Difference, of the given Roots.

Thus, $a \times r^{\frac{1}{n}} \pm b \times r^{\frac{1}{n}} = \overline{a \pm b} \times r^{\frac{1}{n}}$

For, if $r^{\frac{1}{n}} = c$, then $a \times r^{\frac{1}{n}} = ac$, and $b \times r^{\frac{1}{n}} = bc$, theref.

$ac \pm bc = \overline{a \pm b} \times c = \overline{a \pm b} \times r^{\frac{1}{n}} = a \times r^{\frac{1}{n}} \pm b \times r^{\frac{1}{n}}$

3. Also

3. Also the *Sum*, or *Difference*, of any two *Square Roots* is equal to

The Square Root of the Sum, or Difference, between the Sum of the Powers, and twice the Product of their Roots!

Thus, $\sqrt{75} + \sqrt{48} = \sqrt{243}$ or $9\sqrt{3}$

Also $\sqrt{9} + \sqrt{4} = \sqrt{25} = 5$

For suppose $\sqrt{a}$, and $\sqrt{e}$ any given Quantities,

'Tis plain, $\sqrt{a} \pm \sqrt{e} = \sqrt{a \pm e} = \sqrt{a^2 \pm 2ae + e^2}$

NOTE.

The *Operations of Compound Surds* are so easily deduced from what has been said of *Simple ones*, that we need not insist on them.

CHAP. VII.

The Method of Resolving Mathematical Problems.

GENERAL DIRECTIONS.

I. **I**T is absolutely necessary to have a clear and distinct Conception of the Conditions of the Question propos'd to be Resolv'd.

II. Then substitute some Character or Letter, for each Quantity concern'd in the Question.

And because in *Algebraic Operations*, the *known* and *unknown Quantities* are oftentimes very much Complicated; 'tis usual, for easier distinction, to put the first Letters of the Alphabet, for *known*, and the latter for *unknown Quantities*.

But

But if *Quantities* of different *Species* be *Noted* by the first Letter of their *Names* (when it can conveniently be done) the *Expression* will then be so Simple, so readily known and remember'd, as to be of vast help and advantage to the *Imagination*; besides, it shortens the *Process*, yet preserves it free from all obscurity; and at the same time, wonderfully facilitates the *Solution* it self, as the *Ingenious Learner* will soon find by *Practice*.

III. *By due reasoning from the Conditions of the Question, let the Quantities concerned therein be justly Stated, and carefully compared; so that their relation to one another may appear, and the difference which renders them unequal, be discovered; and consequently the same Thing found expressible two ways or brought into an Equation, or into several Equations independent on each other.*

N O T E.

1. If there are as many *Equations* given, as required.

Then the *Question* has a determinable Number of *Solutions*;

Because, each *Quantity* therein has but one value.

As suppose a *Question* proposed concerning the *Age* of 3 *Persons*, was thus Condition'd, viz.

The 2d. is 7 *Years* older than the 1st. The *Age* of the 3d. is Triple that of the 1st. and 2d. And the *Sum* of their *Age* is 68 *Years*; required the *Age* of each?

Put x for the *Age* of the 1st.

Then $3x + 7x + 21$ is the *Age* of the 1st. 2d. and 3d.

And $x + x + 7 + 6x + 21$, or $8x + 28 = 68$, by the conditions of the *Question*.

R

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So that here is but one Equation given, and one required.

2. If there are more Equations given than required.

Then the Excesses are superfluous, and may sometimes happen to be contrary to, or inconsistent with the rest; and by that means, not only limit, but render the Question incapable of any Solution.

3. If there are more Equations required than given.

Then the Question is Imperfectly determined; Because capable of an infinite number of Answers.

As suppose a Question propos'd concerning the Age of three Persons, was thus condition'd, viz.

The Age of the first is equal to the square of the Age of the second?

And the Sum of the Age of the 1st. and 2d. is equal to the Age of the 3d.

Required the Age of each.

Put x , y , and z , for the Age of the 1st. 2d. and 3d. Person.

Then $x = yy$, and $x + y = z$, by the Conditions of the Question.

So that here are two Equations given, and three required:

Therefore the Question is Imperfectly determin'd;

For any one of these Quantities may be assumed at Pleasure.

IV. If the Question when stated, is found to have a determinable Number of Solutions.

Then the Equation directly drawn from the Conditions of the Question, (because at that time the known and unknown Quantities, are for the most part very much Compounded and mix'd together on each side thereof) must be reduc'd into another, by Equal Augmentation and Diminution, where

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where { only one Power } of the unknown Quantity { is }
 { different Powers } { are }
 found equal to some known ones; which is call'd a
Simple or Pure { Equation; And is then prepared for
Compound or Affected {
Solution, and distinguish'd by the Name of the Index of
 the Highest Power of the Quantity sought.

V. Let the Equation be cleared from all Fractions and Surd Roots, as also the highest unknown Power from any Coefficient.

VI. Let all the Terms on the Right-side be Transposed to the Left, so that the whole be made equal to nothing; And let the several like Terms be brought into one, by prefixing their Signs and Coefficients; placing the highest unknown Power first, the rest succeeding in Order, and the Absolute known Quantity last.

REDUCTION of EQUATIONS.

I. By Transposition.

That is, by Equal Addition, if the Quantity be Negative, and by Equal Subduction, if Affirmative;

And is performed by transferring the Quantity to the other side of the Equation, with a contrary sign.

$$\begin{array}{r|l} \text{If } x - 10 = 40 & \text{If } x + 10 = 40 \\ \text{Add } + 10 & \text{Sub. } - 10 \\ \hline \text{Then } x - 10 + 10 = 40 + 10 & x + 10 - 10 = 40 - 10 \\ \text{i. e. } x = 50 & \text{Or } x = 30 \end{array}$$

$$\begin{array}{r|l} \text{If } x^3 - 3ax^2 = b^3 - b^2x + 2ax^2 & \\ - 2ax^2 + b^2x & + b^2x - 2ax^2 \\ \hline \text{Then } x^3 - 5ax^2 + b^2x = b^3 & 0 \quad 0 \end{array}$$

The Reason of this Operation is plain from Axiom 1.

R 2

II. By

II. By Equal Multiplication.

$$\text{If } \frac{r}{a} = b. \text{ If } \frac{r^3 + 3a^2}{c} + n + a = r + a.$$

$$\text{Mult. by } a \quad a \quad \text{Mult. by } c \quad c$$

$$\text{Then } r = ab. \quad \text{Then } r^3 + 3a^2 + cn = cr.$$

$$\text{If } \frac{r^3}{4} + ar^2 - \frac{cr^3}{a} = ac^2.$$

$$\text{Then } ar^3 + 4a^2r^2 - 4cr^2 = 4a^2c^2.$$

The Reason is evident from Axiom 2.

And by this means Equations are cleared from Fractions.

III. By Equal Division.

$$\begin{array}{l|l|l} \text{If } ax = c & ar + er = cb & 5y + 20 = 60 \\ \text{Divid. by } a & \frac{a}{a+c} \frac{cb}{a+c} & \frac{5}{5} \end{array}$$

$$\text{Then } x = \frac{c}{a} \quad r = \frac{cb}{a+c} \quad y = \frac{60 - 20}{5} = 8$$

$$\text{If } r^4 + 2ar^3 = b^3r^3, \text{ then } r^2 + 2ar = b^3.$$

$$\text{If } r^4 - 2ar^3 = b^3r, \text{ then } r^3 - 2ar = b^3.$$

The Reason of this Operation is also evident from Axiom 2.

VI. By Equal Involution.

By means of which Equations are clear'd from surd Quantities.

$$1. \text{ If } \sqrt{ax} + b = c, \text{ or } \sqrt{ax} = c - b.$$

$$\text{Then } ax = c^2 - 2cb + b^2, \text{ by Squaring each side,}$$

2. If

2. If $x - a = \sqrt{abx - a^2c}$; Then,
 $x^3 - 3ax^2 + 3a^2x - a^3 = abx - a^2c$, by *Cubing*.
 3. If $x^3 + -x\sqrt{a} + \sqrt{b} = 0$, put $\sqrt{a} = m$, and $\sqrt{b} = n$
 Then $x^3 + -mx + n = 0$. And $m = \frac{x^3 + n}{x}$ by *Tr. & Di*.
 Theref. $m^2 x^2 = x^6 + 2nx^3 + n^2$, by *Mult. and Invol*.
 And $-n = \frac{x^6 - m^2 x^2 + n^2}{2x^3}$ by *Transf. and Divis*.
 Therefore, $x^{12} + -2m^2 x^8 - 2n^2 x^6 + m^4 x^4 - 2m^2 n^2 x^2 + n^4 = 0$, by *Mult. Invol. and Transf*.

V. By Equal Evolution.

1. If $zz = 25$, then $z = \sqrt{25} = 5$.
 2. If $x^2 + 2xa + a^2 = c^2$, then $x + a = c$
 3. If $z^4 + 2az^3 + a^2z^2 = m^4n^2$, Then,
 $z^2 + az = m^2n$, by *Extr. the Sq. Root*.
 4. If $z^6 - 3az^5 + 3a^2z^4 - a^3z^3 = 8m^3n^3$,
 Then $z^2 - az = 2mn$, by *Extr. the Cube Root*.
 5. If $x^2 + 2xa = c^2$, then by *Extr. the Sq. R.* of the
unknown side, we have $x + a$, with $-a^2$ remaining;
 Therefore, $x + a (=x^2 + 2xa + a^2)^{\frac{1}{2}} = c^2 + a^2)^{\frac{1}{2}}$.
 And if $x^2 - 4ax = c^2$, then the *Sq. R.* of the
unknown side will be $x - 2a$, with $-4a^2$ remain-
 ing; therefore,

$$x - 2a (=x^2 - 4ax + 4a^2)^{\frac{1}{2}} = c^2 + 4a^2)^{\frac{1}{2}}.$$

Therefore this Operation may be performed, by *Ad-
 ding the Square of half the Coefficient to each side of such
 Equation*; and is called (not improperly) *Compleating
 the Square*

6. If

6. If $y^4 + 4ay^3 + 4a^2y^2 - c^2y^2 - 2ac^2y + \frac{1}{4}c^4$,
 then the Sq. Root of the unknown file will be $y^2 + 2ay - \frac{1}{2}c^2$, with $-\frac{1}{4}c^4$ remaining; therefore,

$$y^4 + 4ay^3 + 4a^2y^2 - c^2y^2 - 2ac^2y + \frac{1}{4}c^4 \Big|^{1/2}$$

will be equal to $2a^2c^2 + \frac{1}{4}c^4 \Big|^{1/2}$

$$\text{That is } y^2 + 2ay - \frac{1}{2}c^2 = 2a^2c^2 + \frac{1}{4}c^4 \Big|^{1/2}$$

S C H O L I U M.

Whence the Ten first Propositions of the second Book of Euclid appear at Sight:

By putting $s = a + e$, then is $a = s - e$, and $e = s - a$.

P R O P O S I T I O N S.

1. $ns = na + ne$ } by Equal Multiplication.
2. $ss = sa + se$ }
3. $sq = a^2 + ae$, and $se = e^2 + ae$ by Eq. Mult.
4. $ss = a^2 + 2ae + e^2$, by Equal Involution.
5. $\frac{1}{4}ss = \frac{1}{4}d^2 + ae$, for $\frac{1}{4}s^2 - \frac{1}{4}d^2 = ae$.
6. $sd + e^2 = a^2$, for $sd = a^2 - e^2$.
7. $\left\{ \begin{array}{l} 2sa + e^2 = s^2 + a^2 \\ 2se + a^2 = s^2 + e^2 \end{array} \right\}$ for $\left\{ \begin{array}{l} e^2 = s^2 - 2sa + a^2 \\ a^2 = s^2 - 2se + e^2 \end{array} \right.$
8. $\left\{ \begin{array}{l} (s+a)^2 = s^2 + 2sa + a^2 = 4sa + e^2 \\ (s+e)^2 = s^2 + 2se + e^2 = 4se + a^2 \end{array} \right\}$ by Preced.
9. $\frac{1}{4}s^2 + \frac{1}{4}d^2 = a^2 + e^2$, for $(s+e)^2 + (s-e)^2 = 2a^2 + 2e^2$
10. $s^2 + d^2 = 2a^2 + 2e^2$, by Preceding.

And by the like Comparisons most of those Propositions given by Viesa, Oughtred, and others, with innumerable more of this kind, of excellent use in the various Parts of the Mathematics, are deduced with great Facility.

The

The Derivation and Composition of EQUATIONS.

Since any Quantity composed of Parts may be reduc'd into Parts, it must evidently follow, That the Original Components or Roots of all Equations, may be either Affirmative, Negative, Mix'd, or Imaginary.

And because in all prepared Equations, any one of its constitutive Roots may be put for the unknown Quantity;

Therefore, all prepared Equations are Really, or Imaginarily constituted by the Products of so many Affirmative Roots with Negative Signs, or Negative Roots with Affirmative Signs, connected to the unknown Quantity, as is denoted by the Index of its Highest Power: thus in

The Origination of Quadratic Equations:

1. If $\begin{cases} x = +a \\ x = +e \end{cases}$ then $\begin{cases} x - x = 0 \\ x - x = 0 \end{cases}$

Therefore
$$\begin{array}{r} xx - ax + ae = 0 \\ -ex \end{array}$$

Put $s = a + e$, then $x^2 - sx + ae = 0$

2. If $\begin{cases} x = -a \\ x = -e \end{cases}$ then $\begin{cases} x + a = 0 \\ x + e = 0 \end{cases}$

Therefore
$$\begin{array}{r} x^2 + ax + ae = 0 \\ +ex \end{array}$$

Put $s = a + e$, then $x^2 + sx + ae = 0$

3. If

3. If $\left\{ \begin{matrix} x = +a \\ x = -c \end{matrix} \right\}$ then $\left\{ \begin{matrix} x - a = 0 \\ x + c = 0 \end{matrix} \right\}$

Therefore
$$\begin{array}{r} xx - ax - ac = 0 \\ + cx \end{array}$$

Put $d = a \cup c$; if $+a$ be $\left\{ \begin{matrix} \text{greater} \\ \text{less} \end{matrix} \right\}$ than $-c$.

Then $xx \mp dx - ac = 0$

Therefore, all *Quadratic Equations* are reducible to one of these *Forms*.

1. $xx \mp dx + ac = 0$

2. $xx \mp dx - ac = 0$

The *Origination* of any *Cubic*, *Biquadratic*, or higher *Equation*, may after the same manner be easily deriv'd.

From which *Comparison* 'tis evident that:

I. In every *Prepared Equation* Really constituted, which has, or is supposed to have, all its *Terms*;

The *unknown Quantity* has so many *Roots* or *Values*, as are the *Dimensions* of its highest *Power*; whereof so many are *Affirmative* as the *Signs* in *Order* have *Changes*, and the rest are *Negative*.

II. In such *Prepared Equations*, by changing the *Signs* in *even Places*;

1. The *Affirmative Roots* are made *Negative*, and the *Negative Roots* *Affirmative*.

Also, the *Coefficients* of the 2d. *Term* is the *Sum* of all the *Roots* so sign'd.

And the *Coefficient* of the 3d, 4th, 5th, &c. *Term*, is the *Sum* of the *Products* made of the *Roots* taken by 2's, 3's, 4's, &c.

But

But the Number of Products, whose Sum is to be taken in the 3d, 4th, 5th, &c. Term is the unciæ of the 3d, 4th, 5th, &c. Term of a Binomial rais'd to the Dimension of this Equation.

Therefore the Last Term or Absolute known Quantity is only the Product of all the Roots.

2, Hence, If all the Negative Products made of the Roots taken by 2's, 3's, 4's, &c. (Secluding their Signs), are equal to all the Affirmative ones, (tho' not respectively one to another) then the 3d, 4th, 5th, &c. Term is wanting.

And the contrary, If these are wanting, those must be equal.

PROBLEM I.

To Increase, or Diminish the value of the unknown Roots of an Equation, by any given Quantity.

RULE.

Instead of the unknown Root, Substitute another Less, or Greater than it, by the Quantity given.

Exam. 1. To Augment the Roots of this Equation, $xx + ax - bb = 0$, by the Quantity c .

Put $x = r - c$, then you will have this Equation

$rr + ar - 2cr - ac + cc - bb = 0$, whose Roots exceed those of the former, by the Quantity c .

Exam. 2. To Diminish the Roots of this Equation, $x^2 + ax - bb = 0$, by the Quantity c .

Put $x = r + c$, then you will have this Equation

$r^2 + ar + 2cr + ac + cc - bb = 0$, whose Roots are less than those of the Equation propos'd, by the Quantity c .

S

COROLL

COROLLARY 1.

To take away any one *Term*, except the *First*, from any given *Equation*.

Let the *Equation* be $x^4 - ax^3 + bx^2 - cx + d = 0$.

Put $y + n = x$. Then

$$\begin{array}{rcccccl} y^4 + 4ny^3 + 6n^2y^2 + 4n^3y + n^4 & = & + & x & x & x & x \\ - a & - & 3an & - & 3an^2 & - & an^3 = - axxx \\ + b & + & 2bn & + & bn^2 & = & + bxx \\ & & - c & - & cn & = & - cx \\ & & & & + d & = & + d \end{array}$$

Now 'tis plain, That any *Term*, except the *First*, may be taken away from this *Equation*, because n was taken at Pleasure.

So that by putting $4n - a = 0$, or $n = \frac{1}{4}a$, the 2d *Term* must vanish;

And putting $6n^2 - 3an + b = 0$, the 3d *Term* will also vanish:

After the same manner any other *Term* in this, or any other *Equation* may be destroy'd.

Therefore, in taking away the 2d *Term* of any *Equation*; Let the *Index* of the highest unknown *Power* be m , and if the *Coefficient* of the 2d *Term* be $\pm a$, substitute $y \mp \frac{a}{m}$ in the room of x every where, then the 2d *Term* of that *Equation* will be destroy'd.

COROLLARY 2.

To add to an *Equation* any *Term* that is wanting.

Suppose the *Equation* was $x^4 + c^3x - d^4 = 0$

Put

Put $x = y - c$, then,

$$\begin{array}{r} y^4 - 4cy^3 + 6c^2y^2 - 4c^3y + c^4 = +xxxx \\ + c^3y - c^4 = +c^3x \\ - d^4 = -d^4 \end{array}$$

$$y^4 - 4cy^3 + 6c^2y^2 - 3c^3y - d^4 = 0.$$

PROBLEM 2.

To Multiply, or Divide the Roots of an Equation,
by any given Quantity.

RULE.

Multiply, or Divide each Term respectively, by a Rank of
Continual Proportionals from 1, whose Ratio is the Quan-
tity given.

Exam. 1. To Multiply the Roots of this Equation,

$$x^3 + ax^2 - b^2x = ccc, \text{ by } 4;$$

Mult. by 1 4 16 64 then,

We have $x^3 + 4ax^2 - 16b^2x = 64c^3$, an Equation
whose Roots are Quadruple those of the former.

COROLLARY 1.

Hence, an Equation may be clear'd from Fractions; By
Multiplying its Roots by the Product of the Denominators
of the Fraction.

Thus, if $x^3 - \frac{2}{3}ax^2 + \frac{3}{4}bbx - ccc = 0$.

Mult. by 1 12 144 1728

Product, $x^3 - 8ax^2 + 108bbx - 1728c^3 = 0$.

NOTE.

If the Coefficient of the 2d Term of any Equation be
not the Index of that Equation's Dimension, or some
Multiple thereof, the Solution will be incumbered with
§ 2 Fractions;

Fractions; to avoid which, Multiply all the Coefficients by a Rank of Continual Proportionals from 1, whose Ratio is the Equation's Dimension.

C O R O L L A R Y 2.

Also, *Equations* may by this Rule, sometimes be clear'd from *Surd Quantities*; By Multiplying the Roots of the Equation by the *Surd Quantities* to be clear'd.

$$\begin{array}{r} \text{Thus, if } x^4 + 2ax^3 + 8bbx^2 - c^3x + 8d^4 = 0 \\ \text{Mult. by } 1 \quad \sqrt[3]{2} \quad 2 \quad \sqrt[3]{8} \quad 4 \\ \hline \text{Prod. } x^4 + 4ax^3 + 16bbx^2 - 8c^3x + 8d^4 \end{array}$$

Exam. 2. To Divide the Roots of this Equation

$$\begin{array}{r} x^3 + 6ax^2 - 18bbx - 16c^3 = 0, \text{ by } 3; \\ \text{Divide by } 1 \quad 3 \quad 9 \quad 27 \quad \text{then,} \\ \hline \end{array}$$

We have $x^3 + 2ax^2 - 2b^2x - 6c^3 = 0$, an Equation, whose Roots are *1*kinds of those of the former.

C O R O L L A R Y.

Hence also *Equations* are sometimes clear'd from *Surd Quantities*, without being rais'd to higher Powers; By Dividing their Roots by the *Surd Quantity* to be clear'd.

$$\begin{array}{r} \text{Thus, if } x^4 - ax^3 + 2bbx^2 - c^3x + d^4 = 0 \\ \text{Divide by } 1 \quad \sqrt[3]{2} \quad 2 \quad \sqrt[3]{8} \quad 4 \\ \hline \text{Quotient } x^4 - ax^3 + bbx^2 - cx + d^4 \end{array}$$

And

And if $x^3 - a x x \sqrt[3]{2} + b b x \sqrt[3]{32} = c d^2$
 Divide by 1 $\sqrt[3]{2}$ $\sqrt[3]{4}$ 2

 Quotient $x^3 - a x x + 2 b b x = \frac{1}{2} c d d.$

THE SOLUTION OF EQUATIONS.

I. Of Simple or Pure Equations.

That the Reader might know the manner of applying the foregoing *Rules*, we thought it necessary in this Place, to insert some of the easiest *Questions* we could find.

And if the Method of Solution here used be well observed, the young Learner will find no difficulty in resolving those that are more intricate.

1. *Quest.* One being ask'd how old he was, Answer'd, If $\frac{1}{3}$ of my Age be Multiplied by $\frac{5}{8}$ of the same, the Product will be my Age; I demand his Age. Suppose it x .

Then $\frac{x}{20} \times \frac{5x}{8} = \frac{5xx}{160} = x$, by the *Question*;

And $160x = 5xx$, by *Multiplication*:

Therefore, $160 = 5x$, and $x = \frac{160}{5} = 32$, by *Divis.*

Quest. 2. One ask'd a Shepherd to sell him a 1000 Sheep, who Answered, that he could not then; for he wanted of his Demands just as many, half as many, and $72\frac{1}{2}$ Sheep more than he had; How many had he? Suppose it x ,

Then

Then $x + x + \frac{1}{2}x + 72\frac{1}{2} = 1000$, by the Question;
 And $2x + \frac{1}{2}x = 1000 - 72, 5 = 927, 5$, by Transp.

Theref. $5x = 1855$, and $x = \frac{1855}{5} = 371$, fought.

Quest. 3. A Privateer running at the rate of 10 Miles an Hour discovers a Ship 6 Leagues off, making away at the rate of 8 Miles an Hour; I demand in how many hours can the Privateer come in with the Ship. Suppose in x hours.

Then $8x + 18 = 10x$, by the Question;

And $10x - 8x = 18$, by Transposition:

Theref. $x = \frac{18}{10-8} = \frac{18}{2} = 9$ hours fought.

Quest. 4. The Age of two Persons A and B, being 100 Years; the Age of A exceeds that of B, by 40 Years: I demand the Age of each. Suppose them x and y .

Then, $\begin{cases} x + y = 100 \\ x - y = 40 \end{cases} \therefore \begin{cases} x = 100 - y \\ x = 40 + y \end{cases} \therefore y = 30$, and $x = 70$.

Or thus,

Suppose $x =$ Age of A, then $x - 40 =$ Age of B,

And $2x - 40 = 100$, by the Question, theref. $2x = 140$,

Conseq. $x = \frac{140}{2} = 70$, and the Age of B is 30.

Quest.

Quest. 5. *The Persons A, B, C, owe me Money, but I forgot both the Sum and Particulars, yet by comparing some Accounts I have by me, it appears that the Debt of A and B is 47l. of A and C, 71 l. of B and C, 88l. I demand each Man's Debt. Suppose them x, y, and z.*

$$\begin{array}{l} \text{The } \left\{ \begin{array}{l} x + y = 47 : \therefore x = 47 - y \\ x + z = 71 : \therefore x = 71 - z \\ y + z = 88 : \therefore y = 88 - z \end{array} \right\} : \therefore y = z - 24. \\ \text{Theref. } z = 56l. \quad y = 32l. \text{ and } x = 15l. \end{array}$$

Quest. 6. *A Gentleman bought a House, a Park, and a Garden; their prices were at 12, 5, and 1; but the Sum of double the price of the House, triple the price of the Park, and quadruple the price of the Garden, is so much greater than 10000 Pounds, as the Sum of the price of the House and Park is less than 5000 Pounds; I demand what each cost.*

The House cost $12x$, Park $5x$, Garden $1x$ } by the Quest.
Then $24x + 15x + 4x = 43x$
And $43x - 10000 = 5000 - 17x$
Theref. $60x = 15000$, by Transposition,

And $x = \frac{15000}{60} = 250$, by Division,

Theref. the House cost 3000l, Park 1250l. Garden 250l.

N O T E.

That some of the preceding Questions with innumerable more such may be resolv'd much easier, by other Methods Peculiar to each;

But our design is only to acquaint the Reader with General ways of resolving such Questions, whence he may at Pleasure draw variety of Particular ones for his Practice. And

And the Method of expressing each unknown *Quantity* by an unknown *Letter*, we take to be the most *General*; because it serves *universally* in all Cases, and does not in the least appear as if *contriv'd*: and therefore is preferable to any other.

Besides, this way cannot possibly strain the Imagination, nor mislead the Fancy, in *finding* other particular ways of *Notation*: For all the *Reasoning*, it supposes, is no more than what is immediately founded on the *Conditions of the Question*; and all the Deductions are Naturally drawn, only by reducing the *Equation* thus form'd, and substituting in the room of certain *unknown Quantities*, their Equivalents in different Expressions.

Those that are able to judge what Method of proceeding is the most Natural will see that we introduce only that which ought to be known, and deserves attention; the which we have insisted on the more that the Beginner might have clear and distinct Ideas of, and be thoroughly acquainted with the *Principal Foundation* of all *Analysis*, that he might hereafter the easier know how to draw useful Inferences, and make Methodical Applications in the more Abstruse Parts.

For *Analysis* of all other *Methods* is the most conducive to, if not the only means for the Discovery of *unknown Truths*; and not only the *Mathematics*, but in General all other *Sciences* would have bin imperfect without it; and the more they observe its *Laws*, the nearer they arrive to Perfection.

Of

Of Indetermined Questions.

Quest. 7. A Composition is to be made of three Ingredients, the whole being 10 Pound Weight, the particulars are worth 2s. 4s. and 8s. the Pound; 'tis required to mix them so, as that the Pound may be afforded for 5s. How much of each must be taken? Suppose x, y, z .

Then $x + y + z = 10$ by the Conditions of the Quest.
And $2x + 4y + 8z = 50$

Here 'tis plain, the Question is Indetermined; because there are three unknown Quantities, and but two Equations.

Therefore let the Limits of any one of the unknown Quantities, suppose z , be determined, which may be done after this manner.

Since $x = 10 - y - z = \frac{50 - 4y - 8z}{2}$ by Transp.

And $y = 15 - 3z$, Therefore $z < 5$.

Also $x (= 10 - 15 + 3z - z) = 2z - 5$. $\therefore z > 2\frac{1}{2}$

Whence these Answers in whole Numbers.

z	x	y
3	2	6
4	3	3

Quest. 8. A Refiner has 8 Ingots of Silver of different fineness, viz. of 4, 6, and of 10 Ounces fine, of which he would mix 20 pound weight, so as to make it 8 Ounces fine; how much must he take of each sort? Suppose x, y , and z .

T

Then

Then $x + y + z = 20$ } by the Question ;
 And $4x + 6y + 10z = 160$ }

$$\text{Theref. } x = 20 - y - z = \frac{160 - 6y - 10z}{4}.$$

And $y = 40 - 3z$, therefore $z < 14$.

Also $x (= 20 - 40 - 3z - z) = 2z - 20 : \therefore z > 10$.

Whence these *Answers in whole Numbers.*

z	y	x
11	7	2
12	4	4
13	1	6

Quest. 9. *A Grocer having 4 sorts of Sugar, at 12 d. 8 d. 6 d. and 4 d. a Pound, desires to have a mixture of a 100 weight made out of them, so as that it may be afforded at 10 d. a Pound: how many Pounds must be taken of each? Suppose x, y, z, u .*

Then $x + y + z + u = 100$ } by the Quest.
 And $12x + 8y + 6z + 4u = 10 \times 100$ }

Theref. $y + z + u = 100 - x$ } by Transp.
 And $8y + 6z + 4u = 1000 - 12x$ }

Then $4y + 4z + 4u = 400 - 4x$ } from x by $\left\{ \begin{smallmatrix} < \\ > \end{smallmatrix} \right\}$ Coeff.
 And $8y + 8z + 8u = 800 - 8x$ }

Then $\begin{cases} 400 - 4x < 1000 - 12x : \therefore x < 75 \\ 800 - 8x > 1000 - 12x : \therefore x > 50 \end{cases}$

Therefore supposing x (for instance) $= 60$; let it be substituted in the *Equations* which express the *Conditions of the Question.*

Then

Then $60 + y + z + u = 100$

And $720 + 8y + 6z + 4u = 1009$

Theref. $y = 40 - z - u = \frac{280 - 6z - 4u}{8}$

And $z = 20 - 2u$, therefore $u < 10$

But $y (= 40 - \frac{20 - 2u - u}{2} - u) = 20 + u$,

Therefore u has no other Limits, but may be any Number under 10.

Hence, in taking $x = 60$, we have these *Answers* in *whole Numbers*.

x	60	60	60	60	60	60	60	60	60
y	21	22	23	24	25	26	27	28	29
z	18	16	14	12	10	8	6	4	2
u	1	2	3	4	5	6	7	8	9

And by taking $x =$ to some other Number between its respective Limits, more *Answers* in *whole Numbers* may be found.

SCHOLIUM.

From *this* and the foregoing Chapters the several *Practical Rules* in *Common Arithmetic* are naturally deduced.

I. The Single Rule of Three, Direct and Inverse.

Wherein there are three *Terms* given, two of which are of the same *Denomination*.

And in *Questions* falling under this *Rule*, the *Terms* may be disposed in this order, viz.

T 2

The

The *Querying Term* }
 That of the *same kind*, } in the $\left\{ \begin{array}{l} 3d. \\ 1st. \end{array} \right\}$ Place : Then
 And that of *different kind* } $\left\{ \begin{array}{l} 1st. \\ 2d. \end{array} \right\}$

If the $\left\{ \begin{array}{l} \text{greater} \\ 3d. \text{ be } \end{array} \right\}$ $\left\{ \begin{array}{l} \text{less} \\ \end{array} \right\}$ than the 1st. and requires $\left\{ \begin{array}{l} \text{greater} \\ \end{array} \right\}$ that the 4th. should be $\left\{ \begin{array}{l} \text{less} \\ \end{array} \right\}$ than the 2d.

That is, if $\left\{ \begin{array}{l} \text{more} \\ \text{less} \end{array} \right\}$ requires $\left\{ \begin{array}{l} \text{more} \\ \text{less} \end{array} \right\}$

The *Proportion* is *Direct* ; But

If the $\left\{ \begin{array}{l} \text{greater} \\ 3d. \text{ be } \end{array} \right\}$ $\left\{ \begin{array}{l} \text{less} \\ \end{array} \right\}$ than the 1st. and requires $\left\{ \begin{array}{l} \text{less} \\ \end{array} \right\}$ that the 4th. should be $\left\{ \begin{array}{l} \text{greater} \\ \end{array} \right\}$ than the 2d.

That is, if $\left\{ \begin{array}{l} \text{more} \\ \text{less} \end{array} \right\}$ requires $\left\{ \begin{array}{l} \text{less} \\ \text{more} \end{array} \right\}$

The *Proportion* is *Inverse* : And

The *Required Term* is = $\left\{ \begin{array}{l} \frac{2d. \times 3d.}{1st.} \text{ in the } \textit{Direct} \\ \frac{1st. \times 2d.}{3d.} \text{ in the } \textit{Inverse} \end{array} \right\}$ Rule.

But in the *Inverse Rule*, if the *Terms* be placed thus
viz.

The *Querying Term* }
 That of the *same kind*, } in the $\left\{ \begin{array}{l} 1st. \\ 3d. \end{array} \right\}$ Place ;
 That of *different kind* } $\left\{ \begin{array}{l} 3d. \\ 2d. \end{array} \right\}$

Then the *Work* is as in the *Direct Rule*.

C O R O L L A R Y.

When the *Rule of Three Direct* has 1 for the 1st. Term, it is usually called the *Rule of Practice*, from its frequent use and ready Performances in Common Affairs ; for,

1. If

1. If the Price, &c. of the Integer be an Aliquot part of a Pound, Shilling, &c. as,

If 1 yd. : 6 d. :: 420 yd. : w? But 6 d. = $\frac{6}{12} s = \frac{1}{2} s$

$$\therefore w (= \frac{1 s \times 420}{1 s.} = \frac{s}{2} \times \frac{420}{1} = \frac{420}{2} s = 210 s) = 210 s.$$

If 1 yd. : 4 d. :: 63 yd. : w? But 4 d. = $\frac{4}{12} s = \frac{1}{3} s$

$$\therefore w (= \frac{1 s \times 63}{1 s.} = \frac{s}{3} \times \frac{63}{1} = \frac{63}{3} s) = 21 s = 21 s.$$

If 1 yd. : 10 s. :: 34 yd. : w? but 10 s. = $\frac{10}{20} l. = \frac{1}{2} l.$

$$\therefore w (= \frac{10 s. \times 34}{20 s.} = \frac{1 l. \times 34}{2 l.} = \frac{1}{2} \times \frac{34}{1} = \frac{34}{2} l.) = 17 l.$$

If 1 yd. : 2 s. :: 56 yd. : w? but 2 s. = $\frac{2}{20} l. = \frac{1}{10} l.$

$$\therefore w (= \frac{2 s. \times 56}{10 s.} = \frac{1 l. \times 56}{10 l.} = 5 l. + \frac{6 \times 2 s.}{10} = 5 l. 12 s.)$$

2. If the Price, &c. of the Integer is Composed of the Aliquot Parts of a Pound, Shilling, &c. as,

If 1 yd. : 9 d. :: 84 yd. : w? but 9 d. = $(\frac{6}{12} s + \frac{3}{12} s) = \frac{1}{2} s + \frac{1}{4} s$

$$\therefore w (= \frac{84}{2} s + \frac{84}{4} s = 42 s + 21 s = 63 s) = 3 l. 3 s.$$

If 1 yd. : 7 s. :: 264 yd. : w? but 7 s. = $(\frac{5}{20} l. + \frac{2}{20} l.) = \frac{1}{4} l. + \frac{1}{10} l.$

$$\therefore w (= \frac{264}{4} l + \frac{264}{10} l = 66 l + 26 l + 4 s \times 2 s) = 92 l. 8 s.$$

3. Also

3, Also if 1 lb. : n f. :: 1 C. (= 112 lb.) w ?
 Since 112 f. (= $2s + 4d$) = $2s + 1$ groat,
 Therefore $w = n \times \frac{2s + 1g}{112}$ sought,

The manner of Operation in the other Cases of the Rule of Practice cannot but appear evident to them that thoroughly understand what is here already delivered, therefore this alone was thought sufficient; especially, since that Accurate Penman and most Ingenious Accomptant Mr. George Shelley, in his late Supplement to Mr. Wingate's *Arithmetic*, has (amongst variety of very useful Compendious Rules for the ready Solving of such Questions, as daily occur in Common and Ordinary Affairs) particularly insisted on the several parts of the Rule of Practice, the which he that would be farther inform'd would do well to consult.

II. The Compound Rule of Three, Direct and Inverse.

Let the Terms of the Supposition with their corresponding ones of the Demand, be so placed, as to appear directly Stated; then 'twill be either;

$$\begin{array}{ccccccc}
 \text{1} & & \text{2} & & \text{3} \\
 \text{A, B, C,} & \text{A, B, C,} & \text{A, B, C,} \\
 \text{a, b, ,} & \text{a, , c, ,} & \text{, b, c,} \\
 \text{where } c = \frac{C a b}{A B} ; & b = \frac{c A B}{C a} ; & a = \frac{c A B}{C b} ;
 \end{array}$$

For

For

$$1. \left\{ \begin{array}{l} A : C :: a : w \\ B : w :: b : c \end{array} \right\} \therefore AB : C :: ab : c = \frac{C \times b}{AB}$$

$$2. \left\{ \begin{array}{l} C : A :: c : w \\ a : w :: B : b \end{array} \right\} \therefore Ca : A :: cB : b = \frac{cAB}{Ca}$$

$$3. \left\{ \begin{array}{l} C : A :: c : w \\ b : w :: B : a \end{array} \right\} \therefore Cb : A :: cB : a = \frac{cAB}{Cb}$$

EXAMPLE 1.

If the $\left\{ \begin{array}{l} \text{Men} \\ \text{The} \end{array} \right\} \left\{ \begin{array}{l} M \\ m \end{array} \right\}$ in the Time $\left\{ \begin{array}{l} T \\ t \end{array} \right\}$ spend Pounds $\left\{ \begin{array}{l} P \\ p \end{array} \right\}$;

$$\text{Then } MT : P :: mt : p = \frac{Pmt}{MT}$$

$$\text{For } M : m :: P : \frac{Pm}{M}$$

$$\text{And } T : t :: \frac{Pm}{M} : \frac{Pmt}{MT} = p \text{ sought.}$$

EXAMPLE 2.

If a $\left\{ \begin{array}{l} \text{Body} \\ \text{The} \end{array} \right\} \left\{ \begin{array}{l} L \\ l \end{array} \right\}$ Long, $\left\{ \begin{array}{l} B \\ b \end{array} \right\}$ Broad, $\left\{ \begin{array}{l} T \\ t \end{array} \right\}$ Thick, $\left\{ \begin{array}{l} W \\ w \end{array} \right\}$ Weight;

$$\text{Then } LBT : W :: lbt : \frac{Wlbt}{LBT} = w.$$

$$\text{For } L : l :: W : \frac{Wl}{L}$$

$$\text{Also } B : b :: \frac{Wl}{L} : \frac{Wlb}{LB}$$

$$\text{And } T : t :: \frac{Wlb}{LB} : \frac{Wlbt}{LBT} = w \text{ sought.}$$

III. The

III. The Rule of Fellowship.

Example in the Single Rule of Fellowship.

Suppose $\left\{ \begin{matrix} M \\ M \\ m \\ \&c. \end{matrix} \right\}$ put $\left\{ \begin{matrix} L \\ L \\ l \\ \&c. \end{matrix} \right\}$ into the Joint Stock;

Let x be the whole Gain, or Loss:

Required the Gain, or Loss of each of them. Then,

$$L+L+l, \&c. : x :: \left\{ \begin{array}{l} L : \frac{Lx}{L+L+l, \&c.} = M's \\ L : \frac{Lx}{L+L+l, \&c.} = M's \\ l : \frac{lx}{L+L+l, \&c.} = m's \\ \&c. \end{array} \right\} \text{Gain, or Loss.}$$

$$\text{For } L+L+l, \&c. : x :: L+L+l, \&c. : \frac{Lx+Lx+lx, \&c.}{L+L+l, \&c.} = x.$$

Example in the Compound Rule of Fellowship.

Sup. $\left\{ \begin{matrix} M \\ M \\ m \\ \&c. \end{matrix} \right\}$ put in $\left\{ \begin{matrix} L \\ L \\ l \\ \&c. \end{matrix} \right\}$ for the Time $\left\{ \begin{matrix} T \\ T \\ t \\ \&c. \end{matrix} \right\}$; Then,

$$LT+LT+l, \&c. : x :: \left\{ \begin{array}{l} LT : \frac{LTx}{LT+LT+l, \&c.} = M's \\ LT : \frac{LTx}{LT+LT+l, \&c.} = M's \\ l : \frac{ltx}{LT+LT+l, \&c.} = m's \\ \&c. \end{array} \right\} \text{Gain, or Loss.}$$

$$\text{For } LT+LT, \&c. : x :: LT+LT, \&c. : \frac{LTx+LTx}{LT+LT, \&c.} = x.$$

IV. of

IV. Of Reduction and Exchanges.

Ex. 1. If 100 Ells of *Antwerp* = 75 Yards of *London*;
How many Yards of *London* = 27 Ells of *Antwerp*?

$$100 A = 75 L, \therefore 1 A = \frac{75}{100} L, \text{ and } 27 A = \frac{27 \times 75}{100} L = 20 \frac{1}{4} L.$$

Ex. 2. If $\left\{ \begin{array}{l} 35 \\ 3 \end{array} \right\}$ Ells of $\left\{ \begin{array}{l} \text{Vienna} = 24 \\ \text{Lyons} = 5 \\ \text{Antwerp} = 125 \\ \text{Vienna} = 50 \end{array} \right\}$ Ells of $\left\{ \begin{array}{l} \text{Lyons} \\ \text{Antwerp} \\ \text{Frankfort} \end{array} \right\}$
How many —

$$35 V = 24 L, \therefore 1 L = \frac{35}{24} V$$

$$3 L = 5 A, \therefore 1 L = \frac{5}{3} A = \frac{35}{24} V, \therefore 1 A = \frac{3 \times 35}{5 \times 24} V.$$

$$100 A = 125 F, \therefore 1 A = \frac{125}{100} F = \frac{3 \times 35}{5 \times 24} V, \text{ Therefore,}$$

$$1 F = \frac{100 \times 3 \times 35}{125 \times 5 \times 24} V, \therefore 50 F = \frac{50 \times 100 \times 3 \times 35}{125 \times 5 \times 24} V = 35.$$

Ex. 3. If $\left\{ \begin{array}{l} 6 \text{ lb. of Pepper} \\ 24 \text{ lb. of Ginger} \\ 5 \text{ lb. of Cinnamon} \\ 1 \text{ lb. of Sugar} \end{array} \right\}$ is worth $\left\{ \begin{array}{l} 12 \text{ lb. of Ginger} \\ 4 \text{ lb. of Cinnamon} \\ 60 \text{ lb. of Sugar} \\ 6 \text{ Pence Sterling} \end{array} \right\}$
How many — Pence is — 12 lb. of Pepper worth?

It

Since

Since,

$$\begin{aligned} 6P = 12G : \cdot 1G = \frac{6}{12}P \\ 24G = 4C : \cdot 1G = \frac{4}{24}C \end{aligned} \quad \therefore 1C = \frac{24 \times 6}{12 \times 4}P.$$

$$5C = 60S : \cdot 1C = \frac{60}{5}S = \frac{24 \times 6}{12 \times 4}P : \cdot 1S = \frac{24 \times 6 \times 5}{12 \times 4 \times 60}P.$$

$$1S = 6d = \frac{24 \times 6 \times 5}{12 \times 4 \times 60}P : \cdot 1P = \frac{12 \times 4 \times 60 \times 6}{24 \times 6 \times 5}d.$$

$$\text{And } 112P = \frac{12 \times 4 \times 60 \times 6 \times 112}{24 \times 6 \times 5}d = 2688d.$$

That is, 112lb. of Pepper is worth 11l. 4s. Sterling.

V. The Rule of Alligation.

This we shall wholly omit as Imperfect, because the Questions falling under it are *Indetermined ones*, which this Rule, as commonly delivered, cannot *fully Solve*; tho' it may perhaps give one or more *True Answers*, yet they may not be those that are required for present occasion; of which several Instances might easily be given, as the Learned Dr. Wallis has done in *Ch. 58. Vol. 2.* of his *Works*.

But as to the *Solutions* of such Questions, we refer the Reader to the Method already given for Solving *Indetermined Questions*, of which the Rule of Alligation is but an *Example*.

VI. The Rule of False Position.

1. Single Position.

Let A represent the *Absolute Number* given;
P any *Supposed Number*;

S, The

S { The Sum produced by P, when order'd according to the Conditions of the Question :

Then $\frac{PA}{S}$ is the Number sought.

This is evident from the Operation it self.

2. *Double Position.*

Let P, p, be the two Positions ;

E, e, { Their respective Errors, when order'd according to the Question ;

n The Number sought : Then,

$$\text{If } \pm E, \pm e, n = \frac{Pe \propto pE}{E \propto e} ; \text{ If } \pm E, \mp e, n = \frac{Pe + pE}{E + e}.$$

For suppose it were required to know what Quantity in- to a would give a n.

$$1. \text{ If } P = n - x \text{ ————— } p = n - z$$

Then $na - xa$ is not $= na$ ————— $na - za$ is not $= na$.

Therefore, $E = na - na - xa$ ————— $e = na - na - za$.

And by Subduction, because the Signs are alike,

$$xa \propto za = E \propto e$$

But $Pe = nza - xza$, and $pE = nxa - zxa$,

Therefore, by Subduction also,

$$Pe \propto pE (= nza \propto xza) = n \times E \propto e$$

Consequently $n = \frac{Pe \propto pE}{E \propto e}$, by Division.

$$2. \text{ If } P = n + x \text{ ————— } p = n - z$$

Then $na + xa$ is not $= na$, $na - za$ is not $= na$

Theref. $E = na + xa - na$, ————— $e = na - na - za$

And by Addition, because the Signs are unlike,

$$xa + za = E + e$$

U 2

But

But $Pe = zan + zax$, and $pE = xan - zax$

Therefore by *Addition* also,

$$Pe + pE (= zan + zax) = E + e \times n.$$

Consequently $n = \frac{Pe + pE}{E + e}$ by *Division*.

II. OF QUADRATIC EQUATIONS.

1. If $xx - ax \left\{ \begin{smallmatrix} - \\ + \end{smallmatrix} \right\} b = 0$, or $xx = ax \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b$; put $y + \frac{1}{2}a = x$.

Then $yy + ay + \frac{1}{4}aa = ay + \frac{1}{2}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b$.

Theref. $yy = \frac{1}{4}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b$, and $y = \sqrt{\frac{1}{4}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b}^{\frac{1}{2}}$

Conseq. $x = + \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b}^{\frac{1}{2}}$

2. If $xx + ax \left\{ \begin{smallmatrix} - \\ + \end{smallmatrix} \right\} b = 0$, or $xx = -ax \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b$; put $y - \frac{1}{2}a = x$.

Then $yy - ay + \frac{1}{4}aa = -ay + \frac{1}{2}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b$

Theref. $yy = \frac{1}{4}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b$, and $y = \sqrt{\frac{1}{4}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b}^{\frac{1}{2}}$

Conseq. $x = - \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} b}^{\frac{1}{2}}$

Therefore if V be put for the *Sign* of any *Term*, and $\wedge$ for the contrary, all *Forms of Quadratics*, with their *Solutions*, will be reduc'd to this one.

If $xx \vee ax \vee b = 0$, then $= \wedge \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa \wedge b}^{\frac{1}{2}}$

These *Solutions* may also be found thus;

$$\text{Form } \begin{cases} 1. rr \pm sr = -a \\ 2. rr \pm dr = +a. \end{cases}$$

Since

Since $\frac{1}{2}s \pm \frac{1}{2}d = \sqrt[4]{e}$, Therefore,

$$\frac{1}{2}s + \frac{1}{2}d \times \frac{1}{2}s - \frac{1}{2}d = \frac{1}{4}s^2 - \frac{1}{4}d^2 = e,$$

And $4e = s^2 - d^2$, Therefore,

$$d = \sqrt{s^2 - 4e}^{\frac{1}{2}}, \text{ and } s = \sqrt{d^2 + 4e}^{\frac{1}{2}}$$

But $\sqrt[4]{\frac{1}{4}d^2}^{\frac{1}{2}} (= \frac{1}{2}d) = \sqrt[4]{\frac{1}{4}s^2 - e}^{\frac{1}{2}}$, and

$$\sqrt[4]{\frac{1}{4}s^2}^{\frac{1}{2}} (= \frac{1}{2}s) = \sqrt[4]{\frac{1}{4}d^2 + e}^{\frac{1}{2}}.$$

Therof. $\left\{ \begin{array}{l} 1. \left\{ \begin{array}{l} +a, +e = +\frac{1}{2}s \pm \sqrt[4]{\frac{1}{4}ss - e}^{\frac{1}{2}} \\ -a, -e = -\frac{1}{2}s \mp \sqrt[4]{\frac{1}{4}ss - e}^{\frac{1}{2}} \end{array} \right. \\ 2. \left\{ \begin{array}{l} +a, -e = +\frac{1}{2}d \pm \sqrt[4]{\frac{1}{4}dd + e}^{\frac{1}{2}} \\ -a, +e = -\frac{1}{2}d \mp \sqrt[4]{\frac{1}{4}dd + e}^{\frac{1}{2}} \end{array} \right. \end{array} \right.$

E X A M P L E S.

1. If $xx - 4x = 12$; then $x = +6$, or -2 .

2. If $xx - 2x = 7$; then $x = 1 \pm 8^{\frac{1}{2}}$, or $1 \pm 2 \times 2^{\frac{1}{2}}$

3. If $x^2 - ax + x = b - c - d$; Then,

$$x = \frac{1}{2}a - \frac{1}{2} \pm \sqrt[4]{\frac{1}{4}a^2 - \frac{1}{2}a + \frac{1}{4} + b - c - d}^{\frac{1}{2}}$$

4. If $x^2 + \frac{a}{b}x = c$, then ;

$$x = \pm \sqrt[4]{\frac{aa}{4bb} + c}^{\frac{1}{2}} - \frac{a}{2b} = \frac{\pm a^2 + 4b^2c}^{\frac{1}{2}} - \frac{a}{2b}$$

$$5. x^2 - abx - cdx + efx - gbx = lmn + pqr - stw$$

$$\text{Then } x = \frac{1}{2}a \pm \sqrt[4]{\frac{1}{4}a^2 + lmn + pqr - stw}^{\frac{1}{2}}$$

6. $x^4 - ax^2 = b$, let $x^2 = y$, then $y^2 - ay = b$, and

$$y = \frac{1}{2}a \pm \sqrt[4]{\frac{1}{4}a^2 + b}^{\frac{1}{2}} : \therefore x = \sqrt[4]{\frac{1}{4}a \pm \sqrt[4]{\frac{1}{4}a^2 + b}^{\frac{1}{2}}}$$

7. $x^6 =$

7. $x^6 - ax^3 = b$, let $x^3 = y$, then $y^2 - ay = b$, and
 $y = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + b}^{\frac{1}{2}}$. $x = \sqrt[3]{\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + b}^{\frac{1}{2}}}$.

8. $nx^2 - ax = b$, then $x = \frac{\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + nb}^{\frac{1}{2}}}{n}$

For $x^2 = \frac{ax}{n} + \frac{b}{n}$, $\therefore x = \sqrt{\frac{\frac{1}{2}a}{n} + \frac{\frac{1}{4}a^2}{nn} + \frac{b}{n}}$

That is, $nx = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + nb}^{\frac{1}{2}}$, theref. $x = \frac{1}{n} \left(\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + nb}^{\frac{1}{2}} \right)$.

9. $nx^2 + 2abx = cdn$, then $x = \frac{\pm a^2b^2 + n^2cd^{\frac{1}{2}} - ab}{n}$

10. $n^2x^2 - m^2x^2 - a^3x = -c^4$, then

$x = \frac{\frac{1}{2}a^3 \pm \sqrt{\frac{1}{4}a^6 - n^2c^4 + m^2c^4}^{\frac{1}{2}}}{nn - mm}$

11. $5xx - 4x = 3$, then $x = \frac{2 \pm \sqrt{19}}{5}$

12. $nnx^4 + dx^2 - ex^2 = +a^6 - b^6 + c^6$, let $d-e=z$.

Then $x = \sqrt[4]{\frac{\frac{1}{2}z \pm \sqrt{\frac{1}{4}z^2 + \frac{a^6 - b^6 + c^6}{nn}}^{\frac{1}{2}}}{n}}$

Quest. 1. A Man bought a Horse, which he sold again for 24 l. and found he had gain'd as much per Cent. as the Horse cost him; I demand what the Horse cost at first. Suppose it cost y Pounds.

Let $24 = a$, $100 = b$,

Then $y : a - y :: b : \frac{ba - by}{x} = y$, by the Quest.

Therefore

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Therefore $yy + by = ba$, by *Mult.* and *Transp.*

Conſeq. $y = \sqrt{ba + \frac{1}{4}bb} - \frac{1}{2}b$.

Or, $y = \sqrt{2400 + 2500} - 50 = 20$ l. fought.

Queſt. 2. *A Vintner bought a parcel of Clarret and Rheniſh Wine, which together coſt him 120 l. as for the ſeparate prices, 'tis only known that their Product was equal to 2700 l. Required what each coſt ?*

The Clarret coſt c

Then Rheniſh coſt $120 - c$

And

$$120c - cc = 2700$$

} by the Queſtion.

Theref. $c = \frac{120 \times 120}{4} - 2700 \Big|^{1/2} + \frac{120}{2} = 90$ fought.

That is, The Clarret coſt 90 l. Rheniſh 30 l.

Queſt. 3. *Having bought a Quantity of Goods, viz. Pepper Cloves, Ambergreice, their prices were forgot, but 'tis known that they were continually Proportional, and that the Pepper and Cloves coſt 10 l. the Ambergreice 24 l. more than the Cloves; 'tis required what each coſt.*

The Pepper coſt p , then Cloves coſt $10 - p$.

Then $p : 10 - p :: 10 - p : \frac{100 - 20p + pp}{p}$

And $\left(\frac{100 - 20p + pp}{p} - 10 - p = \right) \frac{100 - 30p + 2pp}{p} = 24$ } by the Queſt.

Therefore $pp - 27p = -50$, by *Mult. Transp.* and *Diviſ.*

And $p = \frac{27}{2} - \frac{729 - 200}{4}^{1/2} = 2$.

Therefore Pepper coſt 2 l. Cloves 8 l. Ambergreice 32 l.

III. OF

III. OF CUBIC EQUATIONS.

All Cubic Equations, whose second Term is destroyed, may be reduced into these Forms $xxx \pm ax - b = 0$, That is, $xxx \pm ax = +b$.

Because in those where the Absolute known Quantity is Negative, there needs no more than making the Roots which were Affirmative in those, Negative in these.

$$\text{If } x^3 \pm ax - b = 0, \text{ then } x^3 \pm a - \frac{b}{x} = 0.$$

And since the $\left\{ \begin{smallmatrix} \text{Diff.} \\ \text{Sum} \end{smallmatrix} \right\}$ of any two like Cubes, is exactly divisible by the $\left\{ \begin{smallmatrix} \text{Diff.} \\ \text{Sum} \end{smallmatrix} \right\}$ of their Roots; Therefore;

Suppose, $b = m^3 \mp n^3$, and $x = m \mp n$, then the Equation will be $\mp 3nm \pm a = 0$, theref. $n = \frac{a}{3m}$

$$\text{And } m^3 \mp (n^3) = \frac{a^3}{27m^3} = b, \text{ then } m^6 - b m^3 = \pm \frac{1}{27} a^3$$

$$\text{Theref. } m^3 = \frac{1}{2} b + \sqrt[3]{\frac{1}{4} b b \pm \frac{1}{27} a^3}, \text{ but } m^3 \pm n^3 = b,$$

$$\text{Then } b = \frac{1}{2} b + \sqrt[3]{\frac{1}{4} b b \pm \frac{1}{27} a^3} \mp n^3, \text{ Or}$$

$$n^3 = \pm \frac{1}{2} b \mp \sqrt[3]{\frac{1}{4} b b \pm \frac{1}{27} a^3}, \text{ and } m \mp n, \text{ i. e.}$$

$$x = \sqrt[3]{\frac{b}{2}} + \sqrt[3]{\frac{bb}{4} \pm \frac{a^3}{27}} \mp \sqrt[3]{-\frac{b}{2}} \pm \sqrt[3]{\frac{bb}{4} \pm \frac{a^3}{27}}$$

Therefore,

1. If $xxx + ax + b = 0$, Then,

$$x = -\sqrt[3]{\frac{b}{2}} + \sqrt[3]{\frac{bb}{4} + \frac{a^3}{27}} \pm \sqrt[3]{-\frac{b}{2}} + \sqrt[3]{\frac{bb}{4} + \frac{a^3}{27}}$$

2. If

2. If $x^3 - ax + b = 0$, Then

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}$$

Having discovered one of the Roots of an Equation, the rest may be found by Division.

And as for *Equations of bigger Dimensions*, we shall omit the like Methods that might be given for their *Solutions*: because more intricate and tedious for Practice, than the Rules of *Numeral Exegesis* delivered by *Vieta*, *Harriot*, and *Oughtred*; which are yet abundantly improved by the Method of *Infinitely Converging Series*, (an *Universal way of Extracting the Roots of any Equation whatsoever.*) And of this Method we shall in its proper Place, give *General Examples*, whence all possible *Particular ones* may be drawn.

NOTE.

That *Equations of high Dimensions* may oftentimes be advantageously reduced lower, by some of the former Rules, or those that may easily be deduced from them, according to the Methods of *Hudden*, *Merry*, and others; and then the *Solution* will be less troublesome.

X

CHAP.

C H A P. VIII.

Of Arithmetic Progression.

DEFINITION I.

A Continued Arithmetic Proportion; that is, where the Terms do Increase and Decrease by equal Differences, is call'd Arithmetic Progression.

Thus $\left\{ \begin{array}{l} a, a+d, a+2d, a+3d, \&c. \text{ Increasing} \\ a, a-d, a-2d, a-3d, \&c. \text{ Decreasing} \end{array} \right\}$ by d .

S C H O L I U M 1.

But since this Progression is only a Compound of two Series,

viz. of $\left\{ \begin{array}{l} \text{Equals } a, a, a, a, a, \&c. \\ \text{Arith. Prop. } 0, \pm d, \pm 2d, \pm 3d, \pm 4d, \&c. \end{array} \right\}$ &c.

Therefore the most Natural Arithmetic Progression is that which begins with 0;

As $0, \pm d, \pm 2d, \pm 3d, \pm 4d, \pm 5d, \left\{ \begin{array}{l} \text{Increasing} \\ \text{Decreasing.} \end{array} \right.$

S C H O L I U M 2.

In an Arithmetic Progression ;

If $\left\{ \begin{array}{l} a \\ d \\ n \\ l \\ s \end{array} \right\}$ be the $\left\{ \begin{array}{l} \text{First Term} \\ \text{Common Difference} \\ \text{Number of Terms} \\ \text{Last Term} \\ \text{Sum of all the Terms} \end{array} \right.$

Then

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Then any *Three* of these *Terms* being given, the other *Two* are easily found.

And the several *Cases* are reducible into *Ten Propositions*, which are all Solved by the *Two* following *Lemmata*.

LEMMA 1.

In any *Arithmetic Progression*;

$$\text{Thus } 1 : \frac{n}{2} :: a + l : s.$$

$$\text{For } \left\{ \begin{array}{c} a \\ a + d \\ a + 2d \\ a + 3d \\ \text{\&c.} \end{array} \right\} + \left\{ \begin{array}{c} l \\ l - d \\ l - 2d \\ l - 3d \\ \text{\&c.} \end{array} \right\} = \left\{ \begin{array}{c} a + l \\ a + l \\ a + l \\ a + l \\ \text{\&c.} \end{array} \right\}$$

$$\text{Therefore } s + s = \overline{a + l} \times n$$

$$\text{That is, } 2s = \overline{a + l} \times n.$$

$$\text{Consequently, } 1 : \frac{n}{2} :: a + l : s.$$

COROLLARIES.

$$1. a = \frac{2s}{n} - l = \frac{2s - nl}{n} = 2s - nl \times \frac{1}{n}$$

$$2. n = \frac{2s}{a + l} = 2 \times \frac{s}{a + l} = 2s \times \frac{1}{a + l}.$$

$$3. l = \frac{2s}{n} - a = \frac{2s - na}{n} = 2s - na \times \frac{1}{n}.$$

$$4. s = \frac{n}{2} \times \overline{a + l} = \frac{n \times a + l}{2} = \frac{na + nl}{2} = n \times \frac{a + l}{2}.$$

X 2

LEM-

L E M M A 2.

In any Arithmetic Progression;

'Tis $1 : n - 1 :: d : l - a$.For, $a, a + d, a + 2d, a + 3d, a + \overline{n-1} \times d = l$.That is, $\overline{n-1} \times d = l - a$, by *Transpos.*Therefore $1 : n - 1 :: d : l - a$.

C O R O L L A R I E S.

$$1. a = l - \overline{n-1} \times d = l - nd + d.$$

$$2. n = \frac{l-a}{d} + 1 = \frac{l-a+d}{d}.$$

$$3. d = \frac{l-a}{n-1} = l-a \times \frac{1}{n-1}.$$

$$4. l = a + \overline{n-1} \times d = a + nd - d.$$

P R O P O S I T I O N 1.

Given a, d, n ; Required l, s .

S O L U T I O N.

$$1. l = a + nd - d = \frac{2s - na}{n} \text{ by Lem. 2d. and 1st.}$$

Then $na + nnd - nd = 2s - na$, by *Mult.*And $2s = 2na + nnd - nd$, by *Transp.*

$$2. \text{Therefore } s = na + \frac{nnd - nd}{2} \text{ by Division.}$$

P R O Q.

PROPOSITION 2.

Given, a, d, l ; Required, n, s .

SOLUTION.

$$1. n = \frac{l - a + d}{d} = \frac{2s}{a + l} \text{ by Lem. 2d. and 1st.}$$

Then $2ds = ll + ld - a^2 + ad$, by *Mult.*

$$2. \text{Therof. } s = \frac{ll + ld - a^2 + ad}{2d}, \text{ by Division.}$$

PROPOSITION 3.

Given, a, d, s ; Required n, l .

SOLUTION.

$$\text{Since } l = \frac{2s - na}{n} = a + nd - d, \text{ by Lem. 1. and 2.}$$

Therof. $nnd + 2na - nd = 2s$ by *Mult. and Transp.*

$$\text{And } nn + \frac{2a - d}{d}n = \frac{2s}{d} \text{ by Division,}$$

$$1. \text{ Then } n = \frac{da + \frac{1}{4}dd - ad + 2ds|^{\frac{1}{2}} - a}{d} + \frac{1}{2}.$$

$$\text{And becauſe } n = \frac{2s}{a + l} = \frac{l - a + d}{d} \text{ by Lem 1. and 2.}$$

Therof. $ll + dl = 2ds - ad + aa$, by *Mult. and Transp.*

$$2. \text{ Then } l = \frac{2ds - ad + aa + \frac{1}{4}dd|^{\frac{1}{2}} - \frac{1}{2}d, \text{ by Compl. the Squar, and Evolut.}$$

PRO-

PROPOSITION 4.

Given, a, l, s ; Required n, d .

SOLUTION.

$$1. n = \frac{2s}{l+a} = \frac{l-a+d}{d} \text{ by Lem. 1 and 2.}$$

Then $2ds - ld - ad = ll - aa$, by *Mult. and Transf.*

$$2. \text{ Theref. } d = \frac{ll - aa}{2s - l - a} \text{ by Division.}$$

PROPOSITION 5.

Given a, n, s ; Required l, d .

SOLUTION.

$$1. l = \frac{2s - na}{n} = a + nd - d \text{ by Lem. 1. and 2.}$$

Then $nnd - nd = 2s - 2na$ by *Mult. and Transf.*

$$2. \text{ Theref. } d = \frac{2s - 2na}{nn - n}, \text{ by Division.}$$

PROPOSITION 6.

Given a, n, l ; Required d, s .

SOLUTION.

$$1. d = \frac{l-a}{n-1} = l - a \times \frac{1}{n-1}, \text{ by Lemma 2.}$$

$$2. s = \frac{na + nl}{2} = a + l \times \frac{n}{2}, \text{ by Lemma 1.}$$

P R O.

PROPOSITION 7.

Given d, l, n ; Required a, s .

SOLUTION.

1. $a = l - nd + d = \frac{2s - nl}{n}$ by Lem. 2. and 1.

Then $2s = 2nl - nnd + nd$, by Mult. and Transp.

2. Theref. $s = \frac{2nl - nnd + nd}{2}$ by Division.

PROPOSITION 8.

Given d, n, s ; Required a, l .

SOLUTION.

Since $l = a + nd - d = \frac{2s - na}{n}$, by Lem. 2. and 1.

Then $2na = 2s - nnd + nd$, by Mult. and Transp.

1. Theref. $a = \frac{2s - nnd + nd}{2n}$ by Division.

And ſince $n = l - nd + d = \frac{2s - nl}{n}$, by Lem. 2. and 1.

Then $2nl = 2s + nnd - nd$, by Mult. and Transp.

2. Theref. $l = \frac{2s + nnd - nd}{2n}$ by Division.

P R O.

PROPOSITION 9.

Given, d, l, s ; Required a, n .

SOLUTION.

Since $n = \frac{2s}{a+l} = \frac{l-a+d}{d}$, by *Lem. 1.* and 2.

Then $aa - ad = ll + ld - 2ds$, by *Mult.* and *Transf.*

1. Theref. $a = \pm \sqrt{ll + ld - 2ds + \frac{1}{4}dd}^{\frac{1}{2}} + \frac{1}{2}d$,

And because, $a = l - nd + d = \frac{2s - nl}{n}$, by *Lem. 2.* & 1.

Theref. $-nnd + 2nl + nd = 2s$, by *Mult.* & *Transf.*

And $-nn + \frac{2l+d}{d}n = \frac{2s}{d}$ by *Division*.

2. Then $n = \frac{1}{2} + \frac{l - \sqrt{ll + \frac{1}{4}dd + ld - 2ds}^{\frac{1}{2}}}{d}$

PROPOSITION 10.

Given, n, l, s ; Required, a, d .

SOLUTION.

1. $a = \frac{2s - nl}{n} = l - nd + d$ by *Lem. 1.* and 2.

Then $2nl - 2s = nnd - nd$, by *Mult.* and *Transf.*

2. Theref. $d = \frac{2nl - 2s}{nn - n}$ by *Division*.

P R O B.

PROBLEM 1.

To find the Sum of the Powers of any Arithmetic Progression.

PREPARATION.

Suppose n the Index of the Power.

Let each Term of the Progression be raised to each Power, under that whose Sum is sought.

And let the Sum of each Rank so rais'd be multiplied by the Multiple of the like Dimension of a in $a + d |^{n+1}$
Put γ for the Sum of all the Products.

And m for the Multiple of a^n in the Power $a + d |^{n+1}$

SOLUTION.

Then $\frac{a + d |^{n+1} - a^{n+1} + n d a^{n+1} + \gamma}{m}$ is the Sum of any Series of Powers whose Roots are Arithmetically Proportional.

For Suppose the Sum of the Cubes of this Arithmetic Progression, $a, a + d, a + 2d, a + 3d$, was required.

$$1. \ a + d |^3 = a + d |^3 = a^3 + 4a^2d + 6ad^2 + 4ad^3 + d^4$$

And the Sum of this Series is $4a + 6d$

Which Multiply by $4d^3$ (the Multiple of a in $a + d |^{3+1}$)

The Product will be $16ad^3 + 24d^4$

Also the Sum of their Squares is $4a^2 + 12ad + 14d^2$

Which Mult. by $6d^2$ (the Multiple of a^2 in $a + d |^{3+1}$)

The Product will be $24a^2d^2 + 72ad^3 + 84d^4$

Y

Therefore

Theref. $\tau = 24a^2d^2 + 88ad^3 + 108d^4 = \text{Sum of these Prod.}$

To which add $a^4 + 4d^4$ $(= a^{n+1} + nd^{n+1} + \tau)$

Sum is $a^4 + 24a^2d^2 + 88ad^3 + 112d^4 (= a^{n+1} + nd^{n+1} + \tau)$

From $l + d | a^{n+1} = a + 4d |^4 = a^4 + 16a^3d + 96a^2d^2 + 256ad^3 + 256d^4$

Subd. $a^{n+1} + nd^{n+1} + \tau = a^4 * + 24a^2d^2 + 88ad^3 + 112d^4$

Then

$l + d |^{n+1} - a^{n+1} + nd^{n+1} + \tau = 16a^3d + 72a^2d^2 + 168ad^3 + 144d^4$

And $\frac{16a^3d + 72a^2d^2 + 168ad^3 + 144d^4}{(m =) 4d} = 4a^3 + 18a^2d$

$+ 42ad^2 + 36d^3$, the Sum of the Cubes of the given Terms.
Because,

The Cube of $\begin{cases} a & \text{is } a^3 \\ a + d & \text{is } a^3 + 3a^2d + 3ad^2 + d^3 \\ a + 2d & \text{is } a^3 + 6a^2d + 12ad^2 + 8d^3 \\ a + 3d & \text{is } a^3 + 9a^2d + 27ad^2 + 27d^3 \end{cases}$

The Sum is $4a^3 + 18a^2d + 42ad^2 + 36d^3$,
the same with the Quotient found.

'Tis the same in any other Series for any other Power.

COROLLARY I.

Therefore, In a Series of Laterals beginning with 1,
if s be put for the Sum of the n Power thereof, Then,

$l + 1 |^{n+1} - 1 + n + \tau = m s$. or,

$$1. l + 1 |^2 - 1 + n = 2s$$

$$2. l + 1 |^3 - 1 + n + 3s = 3s$$

$$3. l + 1 |^4 - 1 + n + 4s + 6s = 4s$$

$$4. l + 1 |^5 - 1 + n + 5s + 10s + 10s = 5s$$

$$5. l + 1 |^6 - 1 + n + 6s + 15s + 20s + 15s = 6s$$

&c.

COROL.

C O R O L L A R Y 2.

Hence, when $a = 0$, then $n = l + 1$, And

$$\begin{aligned} 1. \overline{l+1}^2 - \overline{l+1} &= \overline{l+1} = 2^1 \\ 2. \overline{l+1}^3 - \overline{l+1} + 3 \frac{\overline{l+1}}{2} &= \frac{2l^3 + 3l^2 + l}{2} = 3^2 \\ 3. \overline{l+1}^4 - \overline{l+1} + 6 \frac{2l^3 + 3l^2 + l}{6} + 4 \frac{l^2 + l}{2} &= l^4 + 2l^3 + l^2 = 4^3 \\ &\&c. \end{aligned}$$

Therefore,

$$\begin{aligned} 1. s &= \frac{l^2 + l}{2} = \frac{l+1}{2} l = \frac{n}{2} l. \\ 2. s &= \frac{2l^3 + 3l^2 + l}{6} = \frac{l+1}{3} l^2 + \frac{l+1}{6l} l^2 = \frac{n}{3} l^2 + \frac{n}{6n-6} l. \\ 3. s &= \frac{l^4 + 2l^3 + l^2}{4} = \frac{l+1}{4} l^3 + \frac{l+1}{4l} l^3 = \frac{n l^3}{4} + \frac{n l^2}{4}. \\ &\&c. \end{aligned}$$

C O R O L L A R Y 3.

But if the *Number of Terms* n be ſuppoſed *Infinite*, and g be put for the *greateſt Term or Power*;

Then $s = \frac{ng}{n+1}$, therefore $1 : n + 1 :: s : ng$.

This Proportion will hold, whether n be *Affirmative*, or *Negative*, *Whole*, *Fraſted*, or *Surd Quantity*.

D E F I N I T I O N II.

THe Sums of *Numbers in a Continued Arithmetic Proportion from Unity* are call'd *Figurate or Combinatory Numbers*. Thus,

Y 2

If

If $1, 1+d, 1+2d, 1+3d, 1+4d, 1+5d, 1+6d, &c.$ be an *Arithmetic Progression*; then

$1, 1+1+d, 1+1+d+1+2d, 1+1+d+1+2d+1+3d$ &c. are *Figurate Numbers*.

For the *Units* in each may be disposed into the *Form* of a *Regular Polygon*, whose *Number of Angles* is $d+2$, and whose *Side* (s) is equal to the *Number of Terms* that compose it.

Let f be the *Figurate Number*,
 s its *Side*, or *Number of Terms* composing it.
 a the *Number of Angles* ($= d+2$).

PROBLEM 2.

Case 1. Given s, d ; Required f ?

SOLUTION.

Since f is but the *Sum* of a *Series Arithmetically Proportional* beginning with 1 , by this *Definition*;

$$\text{Theref. } \frac{s+s1}{2} (= \frac{s \times 1 + 1}{2}) = f, \text{ by Lemma 1.}$$

$$\text{But } sd - d + 1 = 1 \quad \text{by Lemma 2.}$$

$$\text{Conseq. } f = \frac{2s + ssd - sd}{2} = s + \frac{ss - s \times d}{2} \text{ by Subst.}$$

Case 2. Given s, a ; Required f ?

SOLUTION.

Since $a = d+2$, or $d = a-2$, by this *Definition*;

$$\text{Theref. } f = \frac{4 + sa - a - 2s \times s}{2} \text{ by Substitution.}$$

PROB.

PROBLEM. 3.

Caſe 1. Given f, d ; Required s ?

SOLUTION.

Since $f = \frac{2s + s s d - s d}{2} = s + \frac{s s d - s d}{2}$ by Prob. 2.

Thereſ. $s s + \frac{2s - s d}{d} = s s + s \times \frac{2 - d}{d} = \frac{2f}{d}$,

Conſeq. $s = \frac{8fd + dd - 4d + 4\sqrt{\frac{1}{2}} + d - 2}{2d}$

Caſe 2. Given f, a ; Required s ?

SOLUTION.

Since $d = a - 2$, by this Definition.

Thereſ. $s = \frac{a^2 - 16a + 16 + 8af - 16f\sqrt{\frac{1}{2}} + a - 4}{2a - 4}$

DEFINITION III.

The Sums of $\left\{ \begin{array}{l} \text{Polygonals} \\ 1^{\text{st}}. \text{Pyramidals} \\ 2^{\text{d}}. \text{Pyramidals} \\ 3^{\text{d}}. \text{Pyramidals} \\ \text{\&c.} \end{array} \right\}$ from Unity $\left\{ \begin{array}{l} 1^{\text{st}}. \\ 2^{\text{d}}. \\ 3^{\text{d}}. \\ 4^{\text{th}}. \\ \text{\&c.} \end{array} \right\}$ are call'd

Pyramidals, having their Names from their Number of Sides.

COROLLARY.

Therefore in a Rank of

$\left\{ \begin{array}{l} \text{Units} \\ \text{Laterals} \\ \text{Triangulars} \\ 1^{\text{st}}. \text{Pyramid} \\ \text{\&c.} \end{array} \right\}$ their Sums are call'd $\left\{ \begin{array}{l} \text{Laterals} \\ \text{Triangulars} \\ 1^{\text{st}} \text{ Pyramid.} \\ 2^{\text{d}} \text{ Pyramid.} \\ \text{\&c.} \end{array} \right\}$ or Rates of the $\left\{ \begin{array}{l} 2^{\text{d}} \\ 3^{\text{d}} \\ 4^{\text{th}} \\ 5^{\text{th}} \\ \text{\&c.} \end{array} \right\}$ Order.

E X.

EXAMPLE.

Units	1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1
Laterals	1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9
Triangulars	1 . 3 . 6 . 10 . 15 . 21 . 28 . 36 . 45
1st. Pyramid.	1 . 4 . 10 . 20 . 35 . 56 . 84 . 120 . 165
2d. Pyramid. &c.	1 . 5 . 15 . 35 . 70 . 126 . 210 . 330 . 495

1. Here 'tis evident, that each *Figurate Number* is the *Aggregate* of the preceding *Series* so far.

2. That each *Figurate Number* is also equal to the *Sum* of the preceding *one*, and that above it.

PROBLEM 4.

To find the *Sum* of such *Series*, or to find any particular *Figurate Number*; by having the *Side* or *Number* of *Terms* composing it given.

1. In *Triangulars* or *Figurates* of the 3d. Order.

(Since every *Triangular Number* is $\frac{ss+s}{2}$, by *Definit. 3*.)

Theref. $\frac{AA+A}{2}$, $\frac{AA+A}{2}$, $\frac{aa+a}{2}$, &c. is a *Series* of *Triangulars*, or *Figurates* of the 3d. Order.

$$\text{But } A + A + a, \text{ \&c.} = \frac{ss+s}{2}$$

$$\text{And } AA + AA + aa, \text{ \&c.} = \frac{s^3+s^2}{3} + \frac{s^3+s^2}{6s}$$

Theref.

Theref. $\frac{AA+AA+aa+A+A+a\&c.}{2} = \frac{s^3+3s^2+2s}{6}$

is the Sum of any Series of *Figurates* of the 3d. Order, or is a *Figurate Number* of the 4th Order, whose side is s .

II. In *Figurate Numbers* of the 4th Order.

Since every *Figurate* of the 4th. Order is $\frac{s^3+3s^2+2s}{6}$ by

the preceding Problem; Therefore the Series is

$$\frac{A^3+3A^2+2A}{6}, \frac{A^3+3A^2+2A}{6}, \frac{a^3+3a^2+2a}{6} \&c.$$

$$\left. \begin{array}{l} 2A + 2A + 2a, \quad \&c. = ss + s \\ 3A^2 + 3A^2 + 3a^2, \quad \&c. = s^3 + s^2 + \frac{s^3+s^2}{2s} \\ A^3 + A^3 + a^3, \quad \&c. = \frac{s^4+s^3}{4} + \frac{s^4+s^3}{4s} \end{array} \right\} \text{by Cor. 2. Prob. 1.}$$

And $\frac{1}{2}$ of the Sum must be $\frac{s^4+6s^3+11s^2+6s}{24}$

which is the Sum of a Series of *Figurates* of the 4th Order, or a *Figurate* of the 5th Order, whose side is s .

Therefore,

$$\left. \begin{array}{l} \frac{s^2+s}{2} = \frac{s+0}{1} \times \frac{s+1}{2} \\ \frac{s^3+3s^2+2s}{6} = \frac{s+0}{1} \times \frac{s+1}{2} \times \frac{s+2}{3} \\ \frac{s^4+6s^3+11s^2+6s}{24} = \frac{s+0}{1} \times \frac{s+1}{2} \times \frac{s+2}{3} \times \frac{s+3}{4} \\ \&c. \end{array} \right\} \&c.$$

is a *Figurate Number* of the 3d. Order; of the Sum of 2d. Order
 of the 4th. a Series of *Figurates* of 3d. Order
 the 5th. of the 4th.

E X.

E X A M P L E.

Given, The Number of Terms, suppose 8, in a Series of Figurates of the 4th Order;

Required, The Sum of that Series; or the 8th. Figurate of the 5th. Order.

Substitute 8 in the room of s , then by the Rule,

$$\frac{1}{1} \times \frac{8+1}{1} \times \frac{8+2}{3} \times \frac{8+3}{4} = 330, \text{ required.}$$

S C H O L I U M.

From what has been here said, we may easily investigate that excellent Theorem of the Illustrious Mr. Newton, for Raising a Binomial, to any given Power.

For let any Binomial $(a+x)$ be raised to any Power, whose Index suppose n , (representing any Number, Affirmative or Negative, Integer or Fraction; The several Powers of that Binomial are,

$$1. a + 1x$$

$$2. a^2 + 2ax + 1x^2$$

$$3. a^3 + 3a^2x + 3ax^2 + 1x^3$$

$$4. a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$$

$$5. a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + 1x^5$$

$$6. a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + 1x^6.$$

&c.

Here 'tis evident at sight, that the Underside of

the	{	1st.	{	Term is a	{	1st.	{	Order, whose Place	{	1		
		2d.				Figurate				2d.	or Side is expres-	$n-0$
		3d.				Number				3d.	sed by	$n-1$
		4th.				of the				4th.		$n-2$
		&c.				&c.				&c.		

Which

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Which being Substituted in the room of s , in the preceding *Theorems* (for finding such *Figurate Numbers*) shall give.

$$\left. \begin{array}{l} 1 \\ 1 \times \frac{n-0}{1} \\ 1 \times \frac{n-1}{1} \times \frac{n-0}{2} \\ 1 \times \frac{n-2}{1} \times \frac{n-1}{2} \times \frac{n-0}{3} \end{array} \right\} \text{ or } \left\{ \begin{array}{l} 1 \\ 1 \times \frac{n-0}{1} \\ 1 \times \frac{n-0}{1} \times \frac{n-1}{2} \\ 1 \times \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \end{array} \right. \\ \&c. \qquad \qquad \qquad \&c.$$

the *Unciæ* or *Coefficients* of the 1st. 2d. 3d. 4th. &c. Term.

Therefore, the *Unciæ* of any *Binomial* $(a+x)$ rais'd to the *Power* whose *Index* is n , will be

$$1 \times \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6}, \&c. i. e. \\ 1. 1 \times \frac{n-0}{1} . 1 \times \frac{n-0}{1} \times \frac{n-1}{2} . 1 \times \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \&c. \\ \text{respectively. Therefore,}$$

$$\begin{aligned} (a+x)^n &= a^n \\ &+ \frac{n-0}{1} a^{n-1} x \\ &+ \frac{n-0}{1} \times \frac{n-1}{2} a^{n-2} x^2 \\ &+ \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} x^3 \\ &+ \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} x^4 \\ &\&c. \end{aligned}$$

2

or

Or putting $q = \frac{x}{a}$, then $a+x=a+a\frac{x}{a}=a+aq=a \times 1+q$

And $\overline{a+aq}^n = a^n$

$$+ \frac{n-0}{1} a^n q$$

$$+ \frac{n-0}{1} \times \frac{n-1}{2} a^n q^2$$

$$+ \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^n q^3$$

$$+ \frac{n-0}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^n q^4$$

&c.

Also putting $A=1$ ft. Term, $B=2$ d. $C=3$ d. $D=4$ th. &c. it will be,

$$\overline{a+aq}^n = a^n + \frac{n}{1} Aq + \frac{n-1}{2} Bq + \frac{n-2}{3} Cq + \frac{n-3}{4} Dq, \&c.$$

Or if the Index be $\frac{m}{n}$, then

$$\overline{a+aq}^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n} Aq + \frac{m-n}{2n} Bq + \frac{m-2n}{3n} Cq + \frac{m-3n}{4n} Dq, \&c.$$

Note, That there are several other ways of *Investigating* this *Important Theorem*, yet none here would be so apposite as this.

And we have insisted the more thereon, because its extensive Use is almost *Infinite*; since 'tis not for the *Discovery* of one *Particular Case* alone that it serves, 'tis not only *Involution*, *Evolution*, *Division by Powers*, or *Radical Quantities*, and the like, as well in *Numbers* as *Species* that it performs; But it even *Comprehends* the *Method of Indivisibles*, the *Arithmetic of Infinities*, the *Doctrine of Series*: and in a word, there is scarce any *Inquiry* so *Sublime* and *Intricate*, or any *Improvement* so *Eminent* and

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and Considerable, in *Pure Mathematics*, but by a *Prudent application* of this *Theorem*, may easily be exhibited and deduced; and that by a *General and Direct Calculation*, in as Perfect a manner as the Nature of the Thing will admit; a few of which we shall Instance.

I. To raise any Tri-nomial, Quadri-nomial, &c. or Infinito-nomial to any given Power.

Suppose the *Infinito-nomial* $a + bz + cz^2 + dz^3$, &c. was to be raised to the Power whose Index is n .

Put $bz + cz^2 + dz^3$, &c. $= x^2$ } in the *Binom. Theor.*
And $bz + cz^2 + dz^3$, &c. $= x^2$ }
&c. Then we have this *Theorem*.

$$\begin{aligned} \overline{a + bz + cz^2 + dz^3, \&c.}^n &= a^n \\ &+ \frac{n}{1} a^{n-1} \times \overline{bz + cz^2 + dz^3, \&c.}^1 \\ &+ \frac{n}{1} \times \frac{n-1}{2} a^{n-2} \times \overline{bz + cz^2 + dz^3, \&c.}^2 \\ &+ \&c. \quad \text{that is} \end{aligned}$$

$$\begin{aligned} \overline{a + bz + cz^2 + dz^3, \&c.}^n &= a^n \\ &+ \frac{n}{1} a^{n-1} bz \\ &+ \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 z^2 \\ &+ \frac{n}{1} a^{n-1} c \\ &+ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 z^3 \\ &+ \frac{n}{1} \times \frac{n-1}{1} a^{n-2} bc \\ &+ \frac{n}{1} a^{n-1} d \\ &+ \&c. \quad Z\ 2 \end{aligned}$$

And Multiplying both Sides by z^n , 'twill be

$$\begin{aligned}
 a z^n + b z^{n+1} + c z^{n+2} + d z^{n+3}, \&c. \big|^n = a^n z^n \\
 &+ \frac{n}{1} a^{n-1} b z^{n+1} \\
 &+ \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 z^{n+2} \\
 &\quad + \frac{n}{1} a^{n-1} c \\
 &+ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 z^{n+3} \\
 &\quad + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b c \\
 &\quad + \frac{n}{1} a^{n-1} d
 \end{aligned}$$

&c. as expres'd by the Author thereof, the Excellent Analyst Mr. *Ab. de Moivre*, (in *Philos. Transf.* N. 230.)

Here 'tis manifest, that

1. All the Products, that can be made so as to have the Sum of the Exponents of the Letters composing them equal to some Index of the Power of z , must belong to z^n Power.

. The Number, which expresses how many ways the Letters of each Product may be changed, must be prefix'd to Product.

But

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But by putting $A = 1st. Term, B = 2d, C = 3d, D = 4th.$
&c. then will

$\frac{a + b\gamma + c\gamma^2 + d\gamma^3 \&c.}{a^n}$ be found equal to

$$+ \frac{1nAb}{1a} \gamma$$

$$+ \frac{2nAc + 1n - 1Bb}{2a} \gamma^2$$

$$+ \frac{3nAd + 2n - 1Bc + n - 2Cb}{3a} \gamma^3$$

$$+ \frac{4nAe + 3n - 1Bd + 2n - 2Cc + 1n - 3Db}{4a} \gamma^4$$

$$+ \frac{5nAf + 4n - 1Be + 3n - 2Cd + 2n - 3Dc + 1n - 4Eb}{5a} \gamma^5$$

&c. where the *Law of Continuation* is visible; but the
Application to Practice is somewhat more difficult than the
former.

II. Of the Nature and Construction of Logarithms.

From hence also, the Celebrated Mathematician Mr.
Halley, Savilian Professor of Geometry in Oxford, among
the several admirable Discoveries, and happy Advances,
in Useful Learning, which he obliges the World with,
has (in *Philos. Trans.* N. 216.) drawn a very curious Me-
thod for Constructing Logarithms, not only compre-
hending all the Improvements that Mercator, Gregory,
and others have made by the Help of Geometric Figures,
but shewing with great Accuracy, from the Common Pro-
perties of Numbers, (as most Natural and Agreeable in
Things purely Arithmetical,) how the Logarithms may be
produced to any desired Number of Places, with far more
Ease and Expedition than by any Method known before.

Since

Since these *Logarithms* (Invented by the Lord *Naper*) as Improved by Mr. *Briggs*, are one of the most useful Discoveries in *Arithmetic*, and the *Demonstration* of this *Method of Constructing* them follows properly in this Place, and is so worthy the *Reader's Knowledge*; we have here inserted a Specimen of their *Nature and Construction*, according to Mr. *Halley's Method*.

1. Supposing an *Infinite Number* n of equal *Ratio's* or *Ratiunculae* in a continued *Scale of Proportionals* between the *Terms* of any *Ratio*, as between 1 and $1 + x$ or $1 + x^n$, then $1 + x$ will be the *First Mean* of *Root* of the *Infinite Power* $1 + x^n$, and let x be a *Ratiuncula*, or *Fluxion* of the *Ratio* of 1 to $1 + x$, then

$$\begin{array}{ccccccc} \text{Sc.} & -2 & -1 & 0 & +1 & +2 & \text{Sc.} & +n \\ \text{Sc.} & \frac{1}{1+x|^2} & \frac{1}{1+x|^1} & 1, \frac{1}{1+x|^1} & \frac{1}{1+x|^2} & \frac{1}{1+x|^2} & \text{Sc.} & \frac{1}{1+x|^n} \end{array}$$

where 'tis evident, that any *Index* expresses the *Number* of *Ratiunculae* contain'd in the *Ratio* of 1 to such *Term*.

2. Hence, we may value *Ratio's* by the *Number* of *Ratiunculae* contain'd in each, and may consider them as *Quantitates sui generis*, beginning from the *Ratio* of 1 to 1 = 0, being *Affirmative*, or *Negative*, according to their *Increase* or *Decrease*, above or below *Unity*.

So that *Ratio's* may be to one another as the *Number* of like and equal *Ratiunculae* contain'd between their *Terms*; and their *Duplicate*, *Triple*, &c. contains *Twice*, *Thrice*, &c. that *Number*.

3. And

3. And the Number of *Ratiunculae* between 1 and any Number or the Value (i. e. the *Exponents*) of the Ratio of Unity to any Number, is called the *Logarithm* of that Number, as it's Etymon $\lambda\omicron\gamma\alpha\nu \alpha\pi\omicron\sigma\mu\delta\varsigma$ very properly Imports.

Thus suppose between 1 and 10 an Infinite Number of mean *Proportionals*, expressed by 10000 &c. in infinitum.

Then between 1 and 2, 1 and 3, 1 and 4, 1 and 10; There will be 3010 &c. 4771 &c. 6989 &c. 10000 &c. which are the *Logarithms* of 2, 3, 4, 10, or rather the *Logarithms* of the Ratio of 1 to 2, 1 to 3, 1 to 4, 1 to 10.

So if the Ratio of 1 to 10 }
That of its { Duplicate } contains { 1000 } &c. *Rati-*
 { Triplicate } { 2000 } *unculae*
 &c. { 3000 } &c.

Therefore, *Logarithms* or the Values of Ratio's are in an *Arithmetic Progression*.

4. But because any Infinite Number of means may be taken between the Terms of any Ratio, provided the same Proportion be every where observed; therefore $n \cdot x$ may as well be put for the *Logarithm* of 1 to $1 + x$, i. e. the Sum of the *Ratiunculae* may as well be the Index as the Number of them; then,

$$o. \quad 1 \cdot x, \quad 2 \cdot x, \quad 3 \cdot x, \quad 4 \cdot x, \quad \&c. \quad n \cdot x.$$

$$1, \quad 1 + x, \quad 1 + x^2, \quad 1 + x^3, \quad 1 + x^4, \quad \&c. \quad 1 + x^n$$

Therefore $\frac{1}{1+x} = 1 - x$ (the *Ratiuncula*)

$$\text{And } \frac{1}{1+x} = 1 - x = n \cdot x = L, \quad 1 + x$$

Therefore

$$\text{But } \frac{\frac{1}{n}-0}{1} \times 1 = +\frac{1}{n}$$

$$\frac{\frac{1}{n}-1}{2} \left(\text{or } \frac{1}{2n} - \frac{1}{2} \right) \times +\frac{1}{n} = \frac{1}{2nn} - \frac{1}{2n}$$

$$\frac{\frac{1}{n}-2}{3} \left(\text{or } \frac{1}{3n} - \frac{2}{3} \right) \times -\frac{1}{2n} = \frac{1}{6nn} + \frac{1}{3n}$$

$$\frac{\frac{1}{n}-3}{4} \left(\text{or } \frac{1}{4n} - \frac{3}{4} \right) \times +\frac{1}{3n} = \frac{1}{12nn} - \frac{1}{4n}$$

And nn being *Infinitely Infinite*, therefore that which is divided thereby muſt vaniſh: Conſequently the Coefficients will be

$$+\frac{1}{n}, -\frac{1}{2n}, +\frac{1}{3n}, -\frac{1}{4n}, +\frac{1}{5n}, \&c. \text{ And}$$

$$1+x = 1 + \frac{1}{n}x - \frac{1}{2n}x^2 + \frac{1}{3n}x^3 - \frac{1}{4n}x^4 + \&c. \text{ Hence}$$

$$\frac{1}{n}x + \frac{1}{2n}x^2 + \frac{1}{3n}x^3 + \frac{1}{4n}x^4 + \&c. = 1 - \frac{1}{1+x} = 1 - \frac{1}{1+x} = L \frac{1}{1+x}$$

And ſince the *Infinite Index* (n), may be aſſumed at pleaſure, the ſeveral *Scales of Logarithms* to ſuch *Indices* will be as $\frac{1}{n}$, or *Reciprocally* as ſuch *Indices*. Therefore, if $n = 10000$, &c. as in *Néper's Logarithms*; then $\frac{1}{n}x + \frac{1}{2n}x^2 + \frac{1}{3n}x^3 + \frac{1}{4n}x^4 + \frac{1}{5n}x^5 + \&c. = x = L, \frac{1}{1+x}$.

But if neither of the *Terms* of the *Ratio* be 1, they muſt be reduced into ſuch, wherein one of the *Terms* may be 1: Thus,

If the *Leaſt* be a , the *Greateſt* b ; let $s = b+a, d = b-a$;

$$\text{Then } \left\{ \begin{array}{l} a : b :: 1 : 1+x = \frac{b}{a} \therefore x = \frac{b}{a} - 1 = \frac{b-a}{a} = \frac{d}{a} \\ b : a :: 1 : 1-x = \frac{a}{b} \therefore x = 1 - \frac{a}{b} = \frac{b-a}{b} = \frac{d}{b} \end{array} \right.$$

A a Therefore

Therefore the *Logarithm* of the *Ratio* contain'd between a and b , may be doubly express'd.

Now suppose the *Ratio* of a to b , was divided into that of a to $\frac{1}{2}s$, and of $\frac{1}{2}s$ to b .

$$\text{Then } L, \frac{a}{\frac{1}{2}s} + L, \frac{\frac{1}{2}s}{b} = L, \frac{a}{b}; \text{ for } \frac{a}{\frac{1}{2}s} \times \frac{\frac{1}{2}s}{b} = \frac{\frac{1}{2}s a}{\frac{1}{2}s b} = \frac{a}{b}$$

$$\text{Or } L, \frac{\frac{1}{2}s}{a} + L, \frac{b}{\frac{1}{2}s} = L, \frac{b}{a}; \text{ for } \frac{\frac{1}{2}s}{a} \times \frac{b}{\frac{1}{2}s} = \frac{\frac{1}{2}s b}{\frac{1}{2}s a} = \frac{b}{a}$$

$$\text{And } \left\{ \begin{array}{l} \frac{1}{2}s : a :: 1 : 1 - x = \frac{a}{\frac{1}{2}s} \\ \frac{1}{2}s : b :: 1 : 1 + x = \frac{b}{\frac{1}{2}s} \end{array} \right\} \parallel \left\{ \begin{array}{l} 1 - \frac{a}{\frac{1}{2}s} = \frac{\frac{1}{2}s - a}{\frac{1}{2}s} \\ \frac{b}{\frac{1}{2}s} - 1 = \frac{b - \frac{1}{2}s}{\frac{1}{2}s} \end{array} \right\} = \frac{d}{s}$$

for both *Ratios*; therefore

$$\frac{1}{n} \times \frac{d}{s} + \frac{d^2}{2s^2} + \frac{d^3}{3s^3} + \frac{d^4}{4s^4} + \frac{d^5}{5s^5}, \&c. = A = L, \left\{ \begin{array}{l} \text{of } \text{Rat. betw.} \\ a \& \frac{1}{2}s \end{array} \right\}$$

$$\frac{1}{n} \times \frac{d}{s} - \frac{d^2}{2s^2} + \frac{d^3}{3s^3} - \frac{d^4}{4s^4} + \frac{d^5}{5s^5}, \&c. = B = L, \left\{ \begin{array}{l} \text{of } \text{Rat. betw.} \\ \frac{1}{2}s \& b \end{array} \right\}$$

R U L E 1.

$$\frac{1}{n} \times \frac{2d}{s} * + \frac{2d^3}{3s^3} * + \frac{2d^5}{5s^5}, \&c. = A + B = L, \text{ Rat. of } a \text{ to } b.$$

$$\text{And } A \propto B = L, \frac{\frac{1}{2}s}{b}) \frac{a}{\frac{1}{2}s} (= \frac{ab}{\frac{1}{4}s^2} \text{ or } \sqrt[4]{\frac{ab}{s^2}})$$

$$\frac{1}{n} \times * \frac{2d^2}{2s^2} * + \frac{2d^4}{4s^4} * + \frac{2d^6}{6s^6} \&c. = L, \frac{\sqrt[3]{ab}}{\frac{1}{2}s} \times \frac{\sqrt[3]{ab}}{\frac{1}{2}s}$$

$$\therefore \frac{1}{n} \times \frac{d^2}{2s^2} + \frac{d^4}{4s^4} + \frac{d^6}{6s^6} + \frac{d^8}{8s^8}, \&c. = L, \text{ of the Ratio of the Geometrical to the Arithmetical mean between } (a) \text{ and } (b).$$

But

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But the *Difference* of the *Terms* of the *Ratio*, i. e. $\frac{1}{4} s s$
 $- a b$, or $\frac{1}{4} a^2 + \frac{1}{2} a b + \frac{1}{4} b^2 - a b = \frac{1}{4} a^2 - \frac{1}{2} a b +$
 $\frac{1}{4} b^2 = \frac{1}{4} a - \frac{1}{2} b \big|^2 = \frac{1}{4} d^2 = 1$ (in this *Case*): And put-
 ting $\frac{1}{4} s s + a b = y$, $\therefore$ (since $y = s$, and $d = 1$) it fol-
 lows, that $\frac{1}{n} \times \frac{2}{y} + \frac{2}{3y^3} + \frac{2}{5y^5} + \frac{2}{7y^7} \&c. = L$, of
 the *Ratio* between $\frac{1}{4} s s$ and $a b$, by *Rule 1*. Whence

R U L E 2.

$\frac{1}{n} \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \&c. = L$, of the
Ratio between $\frac{1}{2} s$ and $\sqrt[2]{a b}$. Which *Rule* is of Excellent use
 for finding the *Logarithms* of *Prime Numbers*, having the
Logarithms of the adjoining *Numbers* given.

In making *Briggs's* *Logarithms*, the *Index* (n) must be
 2,3025850, &c. as was hinted before;

For if the *Index* n be 100000000 &c. in *infinitum*,

The *Logarithm* of 10 will be 2,3025850, &c. as in
Naper's;

But that the *Logarithm* of 10 may be 1000000, &c. as
 in *Briggs's*,

The *Index* n must be 2, 3025850, &c.

And this *Index*, i. e. *Naper's* *Logarithm* of 10, may
 be easily found several ways, either by the Number 10
 it self, or by its *Component Parts*: But we shall instance
 only this way.

Since $10 = 2 \times 2 \times 2 \times 1\frac{1}{4} \therefore L_{10} = 3 L_2 + L_{1\frac{1}{4}}$;
 therefore the *Log.* of 2, and the *Log.* of $1\frac{1}{4}$ must be found
 to the *Index* 100000, &c.

A a 2

And

And Nap. L. 2 is $\frac{2}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{5} \times \frac{2}{3} + \frac{1}{7} \times \frac{2}{3}$ &c. R. 1.

$\therefore 3L, 2 = \frac{6}{3} + \frac{1}{3} \times \frac{6}{3} + \frac{1}{5} \times \frac{6}{3} + \frac{1}{7} \times \frac{6}{3}$ &c. by Mult.

Also Nap. L. $1\frac{1}{2}$ is $\frac{2}{9} + \frac{1}{3} \times \frac{2}{9} + \frac{1}{5} \times \frac{2}{9} + \frac{1}{7} \times \frac{2}{9}$ &c. R. 1

But $\frac{2}{9} = \frac{6}{3^2}$, $\frac{2}{9^2} = \frac{6}{3^4}$, $\frac{2}{9^3} = \frac{6}{3^6}$, &c. Therefore.

The Operation stands thus:

$\frac{6}{3} = 2$,	$\frac{1}{3} 2$ 2,
$2 = A$ 222222222222	$A + \frac{1}{3} A$ 2962962962962
$A = B$ 246913580246	B 49382716049
$B = C$ 27434842249	$C + \frac{1}{3} C$ 13064210595
$C = D$ 304831.805	D 338701756
$D = E$ 338701756	$E + \frac{1}{3} E$ 98531420
$E = F$ 37633529	F 2894887
$F = G$ 4181503	$G + \frac{1}{3} G$ 876124
$G = H$ 464612	H 27330
$H = I$ 51624	$I + \frac{1}{3} I$ 8454
$I = K$ 5753	K 273
$K = L$ 637	$L + \frac{1}{3} L$ 85
$L = M$ 71	M 3

Therefore Naper's Log. of 10, is 2,3025850929940

Or the Index (n) for Brigg's Scale of Logarithms; which, by continuing the the Operation, is found to be

2.30258509299404568401799145468436420760110148

8628772976033328, &c. therefore $\frac{1}{n}$ will be

0.43429448190325182765112891891660508229439700

5803666566114454, &c.

EXAM-

EXAMPLE 1.

To find *Briggs's* Logarithm of 2, only to 10 Places.

Note, That the *Index* muſt be aſſumed of a *Figure* or *two* more than the intended *Logarithm* is to have;

therefore, in this *Example*, $\frac{1}{n} = 0,434294481903 = B$,
 $d = 1$, and $s = 3$.

$$\text{And } \beta \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3^2} + \frac{1}{3^2} \times \frac{2}{3^3} + \frac{1}{3^3} \times \frac{2}{3^4} + \frac{1}{3^4} \times \frac{2}{3^5} + \frac{1}{3^5} \times \frac{2}{3^6} + \frac{1}{3^6} \times \frac{2}{3^7} \&c. = \frac{1}{2} L, 2.$$

$$\text{Or } \frac{\beta}{3} + \frac{1}{3} \times \frac{\beta}{3^2} + \frac{1}{3^2} \times \frac{\beta}{3^3} + \frac{1}{3^3} \times \frac{\beta}{3^4} + \frac{1}{3^4} \times \frac{\beta}{3^5} + \frac{1}{3^5} \times \frac{\beta}{3^6} \&c. = \frac{1}{2} L, 2.$$

The Operation ſtands thus:

$\frac{1}{2} \beta = A$	144764827301	$\frac{1}{2} A =$	144764827301
$\frac{1}{3} A = B$	16084980811	$\frac{1}{3} B =$	5361660270
$\frac{1}{9} B = C$	1787220090	$\frac{1}{9} C =$	357444018
$\frac{1}{27} C = D$	198580010	$\frac{1}{27} D =$	28368572
$\frac{1}{81} D = E$	22064445	$\frac{1}{81} E =$	2451605
$\frac{1}{243} E = F$	2451605	$\frac{1}{243} F =$	222873
$\frac{1}{729} F = G$	272400	$\frac{1}{729} G =$	20953
$\frac{1}{2187} G = H$	30266	$\frac{1}{2187} H =$	2017
$\frac{1}{6561} H = I$	3362	$\frac{1}{6561} I =$	197
$\frac{1}{19683} I = K$	373	$\frac{1}{19683} K =$	19
$\frac{1}{59049} K = L$	41	$\frac{1}{59049} L =$	1
Sum = $\frac{1}{2} L, 2$		= 0,150514997826	
		x 2	
Theref. the <i>Logarithm</i> of 2 is		0,3010299956,52	

But the ſame *Logarithm* may yet be obtained much eaſier and ſooner from this Conſideration, viz.

That

That $\frac{1}{2} \Big|^{10} = \frac{1}{1024}$, and $\frac{1000}{1024} \times \frac{1}{1000} = \frac{1}{1024}$;

Therefore $\frac{L, \frac{1000}{1024} + L, \frac{1}{1000}}{10} = \frac{L, \frac{1}{1024}}{10} = L, 1 \text{ to } 2, = L, 2.$

But $L, \frac{1000}{1024} = L, \frac{125}{128} = \beta \times \frac{2}{1} \times \frac{3}{253} + \frac{2}{3} \times \frac{3^3}{253^3} + \frac{2}{5} \times \frac{3^5}{253^5} \&c.$

i.e. $L, \frac{125}{128} = \frac{2\beta d}{s} + \frac{2\beta d^3}{3s^3} + \frac{2\beta d^5}{5s^5} + \frac{2\beta d^7}{7s^7} \&c.$ by R. 1. Let $\frac{d}{s} = q.$

$\frac{1}{1} \times \frac{1}{1} \times 2\beta d = A = 0.01029947387912$

$\frac{1}{3} \times q \times q \times A = B = 48271995$

$\frac{1}{5} \times q \times q \times B = C = 4072$

Sum $L, \frac{1000}{1024} (= L, \frac{125}{128}) = 0.01029995663980$

Add $L, \frac{1}{1000} = 3,00000000000000$

And $\frac{1}{10}$ of that Sum is $0,30102999566398,0 = L, 2.$

EXAMPLE 2.

To find Briggs's Logarithm of 3.

Here $d=2, s=4$, therefore $\frac{d}{s} = \frac{1}{2}$, conseq. by R. 1.

$L, 3 = \beta \times \frac{2}{1} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{8} + \frac{2}{5} \times \frac{1}{32} + \frac{2}{7} \times \frac{1}{128} \&c.$

$\frac{1}{2} L, 3 = \frac{\beta}{2} + \frac{1}{3} \times \frac{\beta}{8} + \frac{1}{5} \times \frac{\beta}{32} + \frac{1}{7} \times \frac{\beta}{128} \&c.$ whence

the several Terms may be readily found by a *Continual Division* of $\frac{\beta}{2}$ by 4, and those again by the *Indices* of the *odd Powers*, each respectively, and twice the Sum of the *Quotes* will be the *Logarithm* of 3 sought.

But

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But the *Logarithm* of 3, is found abundantly ſooner, by *Rule* 2. The *adjoining Numbers* being 2 (=a) and 4 (=b)

$$\text{And } \left\{ \begin{array}{l} \frac{1}{2}s = 3 \\ \sqrt[2]{ab} = \sqrt[2]{4 \times 2} \end{array} \right\} = \left\{ \begin{array}{l} \text{Ari th.} \\ \text{Geom.} \end{array} \right\} \text{mean betw. 2 and 4.}$$

$$\text{And } \frac{1}{4}ss = 9, \text{ and } \frac{1}{4}ss + ab = 9 + 8 = 17 = y.$$

$$\text{But } \frac{\sqrt[2]{4 \times 2}}{\frac{1}{2}s} \times \frac{1}{\sqrt[2]{4 \times 2}} = \frac{1}{\frac{1}{2}s}, \text{ therefore,}$$

$$L, \frac{1}{\sqrt[2]{4 \times 2}} + L, \frac{\sqrt[2]{4 \times 2}}{\frac{1}{2}s} = L, \frac{1}{\frac{1}{2}s} = L, \text{ of 1 to 3, or } L, 3.$$

$$\text{The } L, \frac{\sqrt[2]{4 \times 2}}{\frac{1}{2}s} = \frac{\beta}{17} + \frac{1}{3} \times \frac{\beta}{17} + \frac{1}{5} \times \frac{\beta}{17} + \frac{1}{7} \times \frac{\beta}{17} \text{ &c.}$$

$$\text{And } L, \frac{1}{\sqrt[2]{4 \times 2}} = \frac{L, 4 + L, 2}{2}$$

0,4515449934959
255467342296
294656680
611744
1511

The Sum is the $L, 3$

$$0,477121254719,0$$

And the ſame *Logarithm* may yet be very expeditiously found by means of the *Ratio* of 2^{25} to 5×3^8 (where $d = 37$, and $s = 65573$).

$$\text{For } \frac{2^{25}}{5 \times 3^8} \times \frac{5}{2^{25}} = \frac{1}{3^8}, \text{ Therefore,}$$

$$L, \frac{2^{25}}{5 \times 3^8} + L, \frac{5}{2^{25}} \text{ or } L, \frac{2^{25}}{5 \times 3^8} + L, 2^{25} - L, 5$$

$$\frac{8}{8} = L, 3$$

But

But $L_1 2^{15} - L_1 5 = 3,81647993062$

And $\frac{1}{4} \times 2 \beta d = 0,00049010708$ 1st Step of the Series.

Sum is $3,81697003770 = L_1 3^8$

And $\frac{1}{4}$ thereof is $0.47712124471 = L_1 3$.

Here the first step alone gives the *Logarithm* true to 11 Places.

Such *Contractions*, to expedite the Operation, may also be found for the other *Prime Numbers*, by means whereof the Series is made to Converge wonderfully quick: as the *Logarithm*

$$\begin{array}{c}
 \left\{ \begin{array}{l} 7 \\ 11 \\ 13 \\ 17 \\ 19 \\ 23 \\ 29 \\ 31 \\ 37 \\ 41 \\ \text{&c.} \end{array} \right\} \text{ by the Ratio of } \left\{ \begin{array}{l} 3 \times 2^1 \times 5^1 \\ 2 \times 7^1 \times 10^2 \\ 7 \times 11 \times 5^2 \times 2^6 \\ 13 \times 2^3 \times 5^2 \\ 10^2 \times 17^1 \\ 10 \times 2^5 \times 3^4 \\ 17 \times 23 \times 2^5 \times 7^2 \\ 10 \times 17 \times 19 \times 2^2 \times 3^2 \\ 3 \times 7 \times 17 \times 29 \times 2^4 \\ 7 \times 10 \times 11 \times 17 \times 2^2 \\ \text{&c.} \end{array} \right\} \text{ to } \left\{ \begin{array}{l} 7^4 \\ 3^4 \times 11^2 \\ 3^6 \times 13^2 \\ 3^2 \times 17^2 \\ 3^2 \times 13^2 \times 19 \\ 7^2 \times 23^2 \\ 3^6 \times 29^2 \\ 11^2 \times 31^2 \\ 11^2 \times 37^2 \\ 37^2 \times 41^2 \\ \text{&c.} \end{array} \right\}
 \end{array}$$

EXAMPLE 3.

To find *Briggs's Logarithm* of the *Prime Number* 23, from the 2d. Rule; as perform'd by Mr. *Halley*, in the aforementioned *Philos. Trans.*

The adjoining Numbers are 22 ($= a$) and 24 ($= b$)
 $\therefore d = 2, s = 46, \frac{1}{4}s = 23$, and $\frac{1}{4}s \pm \frac{1}{4}ab$ or 529 ± 528
 $= 1057 = y$.

But

But

$$\frac{\sqrt{ab}}{\frac{1}{2}s} \times \frac{1}{\sqrt{ab}} = \frac{1}{\frac{1}{2}s} \cdot L, \frac{\sqrt{24 \times 22}}{\frac{1}{2}s} + L, \frac{1}{\sqrt{24 \times 23}} = L, \frac{1}{\frac{1}{2}s} = L, 23.$$

And $L, \frac{\sqrt{24 \times 22}}{\frac{1}{2}s} = \frac{\beta}{y} + \frac{\beta}{3y^3} + \frac{\beta}{5y^5} + \frac{\beta}{7y^7} + \frac{\beta}{9y^9} \&c.$ by A. 1.

Since $2 \times 2 \times 3 = 24$, and $2 \times 11 = 22$, therefore

$$3L, 2 + L, 3 = L, 24, \text{ and } L, 2 + L, 11 = L, 22. \text{ But } \frac{L, 24 + L, 22}{2}$$

Or $L, \sqrt{ab} = 1,$	36131696126690612945009172659809
$\frac{1}{2} \times \frac{1}{2} \times \beta = A$	41087462810146814347315886368
$\frac{1}{3} \times \frac{1}{27} \times A = B$	12258521544181829460074
$\frac{1}{4} \times \frac{1}{27} \times B = C$	6583235184376175
$\frac{1}{5} \times \frac{1}{27} \times C = D$	4208829765
$\frac{1}{7} \times \frac{1}{27} \times D = E$	2930

Sum = $L, 23 = 1.36172783601759287886777711225117$

The *Ingenious* may for his Practice, with the same ease continue the *Logarithm* to any *Number of Places*, by taking the *Index* accordingly; tho the design of this *Treatise*, and the narrowness of the *Page*, determin'd our *Example* but to few *Places*; in the Explication of which, we have bin the more large, to the end that the *Logarithm* of any other *Incomposise Number* may be made by the foregoing *Rules* without any farther *Direction*.

N O T E.

1. The *Logarithms* of *Composise Numbers* are found by Adding of the *Logarithms* of their *Factors*; Thus, the *Logarithm* of 6 is the *Sum* of the *Logarithms* of 2 and 3, for $2 \times 3 = 6$.

B b

2. The

So that an indifferent Capacity may hence see how the *Logarithms of Numbers* are made to any *exactness* or *Number of Places*; and therefore, how the whole *Table of Logarithms* may be Calculated anew, or how one already made may be *examined*, with all desirable *Facility* and *Dispatch*.

And as for the *Tables* themselves; 'tis not thought necessary here to insist on their various *Uses*, because they are so evident to them that understand what is already said, and so largely handled and exemplified by most Writers of *Practical Mathematics*.

C O R O L L A R Y.

Hence also, from the Logarithm given, 'tis easy to find what Ratio it expresses.

For $L, 1 \pm x$ (or L) $= 1 \pm \frac{1}{1 \pm x} \therefore 1 \pm L = \frac{1}{1 \pm x}$

Conseq. $|1 \pm L|^2 = 1 \pm x$, that is, equal to

$$1 \pm nL + \frac{1}{2}n^2L^2 \pm \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4 \pm \frac{1}{120}n^5L^5, \&c.$$

Therefore, if *Naper's Log.* be given (since $n = 1000$ &c.)

$$1 \pm x = 1 \pm L + \frac{1}{2} L^2 \pm \frac{1}{6} L^3 + \frac{1}{24} L^4 \pm \frac{1}{120} L^5, \&c.$$

Wherefore, if one of the *Terms* (*a* being the *Least*, and *b* the *Greatest*) of the *Ratio*, whereof *L* is the *Logarithm*, be given, the other is readily found; for in

$$\text{Napier's Log. } \left\{ \begin{matrix} b=a \\ a=b \end{matrix} \right\} \times \frac{1 \pm L + \frac{1}{2}L^2 \pm \frac{1}{6}L^3 + \frac{1}{24}L^4, \&c.}{1}$$

Or

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Or let the next neareſt *Logarithm* be call'd α , if *Leſs*, β if *Greater* than L , (its Num. a , or b) and $L - \alpha$, or $\beta - L = \delta$, then

$\frac{a}{b} \times 1 \pm \delta + \frac{1}{2} \delta^2 \pm \frac{1}{6} \delta^3 + \frac{1}{24} \delta^4 \pm \frac{1}{120} \delta^5$, &c. = N ,
the Number answering to the *Logarithm* L ; where the
Series converges according to the ſmallneſs of δ .

And the 1ſt. *Step* $a + a \delta$, $b - b \delta$, or $a + n a \delta$, $b - n b \delta$,
ſerves for the *Common Tables*.

Alſo $\frac{a}{1 - \frac{1}{2} \delta} - \frac{\frac{1}{2} a \delta^2}{1 - \delta} + \frac{\frac{1}{6} a \delta^3}{1 - \frac{3}{2} \delta}$, &c. = N .

Where the 1ſt. *Step* only, is ſufficient for *Practice*, in
Tables exceeding any yet extant.

$$i. e. N = a + \frac{a \delta}{\frac{1}{n} - \frac{1}{2} \delta}, \text{ and } b - \frac{b \delta}{\frac{1}{n} + \frac{1}{2} \delta}.$$

$$\text{Or } \frac{\frac{1}{n} + \frac{1}{2} \delta \times a}{\frac{1}{n} - \frac{1}{2} \delta}, \text{ and } \frac{\frac{1}{n} - \frac{1}{2} \delta \times b}{\frac{1}{n} + \frac{1}{2} \delta} = N.$$

Thereſe. $\frac{1}{n} \mp \frac{1}{2} \delta : \frac{1}{n} \pm \frac{1}{2} \delta :: \left\{ \frac{a}{b} \right\} : N$. ſought.

II. To Extract the Root of an Infinite Equation.

Suppoſe, for inſtance, the *Infinite Equation* to be

$$x^2 + b x^2 + c x^3 + d x^4, \&c. = a y + \beta y^2 + \gamma y^3 + \delta y^4, \&c.$$

Put $x = A y + B y^2 + C y^3 + D y^4 + E y^5, \&c.$ Then by the 1ſt.

$$\begin{array}{rcll} x^2 = & 4 A x y + 4 B x y^2 + 4 C x y^3 + 4 D x y^4 + 4 E x y^5 & & \\ + b x^2 = & : + b A^2 : & + 2 b A B : & + 4 B^2 : + 2 b B C : \\ & : & : & + 2 b A C : + 2 b A D : \\ + c x^3 = & : & : & + 3 c A^2 B : + 3 c A B^2 : \\ & : & : & + 3 c A^2 C : \\ + d x^4 = & : & : & + 4 d A^3 B : \\ + e x^5 = & : & : & + e A^5 : \end{array} \quad \left. \vphantom{\begin{array}{rcll} x^2 = \\ + b x^2 = \\ + c x^3 = \\ + d x^4 = \\ + e x^5 = \end{array}} \right\} \&c.$$

$$\&c. = a \times y + \beta \times y^2 + \gamma \times y^3 + \delta \times y^4 + \epsilon \times y^5 \&c.$$

Where 'tis evident, by comparing the *Coefficients*, That

$$B b 2$$

$$A =$$

$$A = \frac{a}{a}$$

$$B = \frac{\beta - bA^2}{a}$$

$$C = \frac{\gamma - 2bAB - cA^3}{a}$$

$$D = \frac{\delta - bB^2 - 2bAC - 3cA^2B - dA^4}{a}$$

$$E = \frac{\epsilon - 2bBC - 2bAD - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5}{a}$$

&c. Therefore, by *Substitution*,

$$z = \frac{a}{a} y$$

$$+ \frac{\beta - bA^2}{a} y^2$$

$$+ \frac{\gamma - 2bAB - cA^3}{a} y^3$$

$$+ \frac{\delta - bB^2 - 2bAC - 3cA^2B - dA^4}{a} y^4$$

$$+ \frac{\epsilon - 2bBC - 2bAD - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5}{a} y^5$$

&c. Which is the *Theorem* given for this purpose, by that Ingenious Mathematician Mr. De Moivre, (in *Philos. Trans.* N° 240.)

And the *Observations* made for the Continuance of this series, are hence also manifest: viz.

1. That each Capital Letter is equal to the Coefficient of the Term preceding that where it was first express'd.

2. That the Denominator of each Coefficient must be always *a*.

3. That

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3. That the 1st. Term of the Numerator must be a respective Coefficient in the Series, $\alpha y + \beta y^2$, &c.

4. That the Capital Letters must be Combin'd, as often as the Sum of their Exponents can be made equal to the Index of the Power to which they belong.

5. That there must be so many Capitals in each Term, as are denoted by the Exponents of the small Letter annex'd.

6. That the Capitals of every Member are capable of so many Permutations as are express'd by the Numeral Figures prefix'd.

NOTE 1. The Quantities a, b, c , &c. α, β, γ , &c. are taken at Pleasure; Therefore may each represent an Infinite Series (if need be), or any indetermin'd Quantities; Consequently, if the Equation involve more than two indetermin'd Quantities, such also may easily be deduced from hence.

NOTE 2. This Theorem may be made Infinitely more General, by Substituting y^a, y^b, y^c , &c. in the room of y, y^2, y^3 , &c. and putting general Exponents for particular ones in the other Indeterminates, then proceed as these Exponents require.

And those also may be made Infinitely more General, by observing the foregoing Method.

III. To Investigate General Ways of Extracting the Roots of all sorts of Equations.

Suppose any Equation whatsoever, as,

$$x^n \dots ax^{n-1} \dots bx^{n-2} \dots cx^{n-3} \dots dx^{n-4}, \&c. \dots A=0$$

Whence A is the Absolute known Quantity;

x the Root required;

n the Index of the Highest Power;

a, b, c , &c. the respective Coefficients.

Put k (a known Quantity, taken at Pleasure, tho' the nearer the True Root the better) $\pm u$ (an unknown Quantity) equal to x ; then the Equation will be

$$1 k \pm u|^n - \dots a k \pm u|^{n-1} - b k \pm u|^{n-2}, \&c. \dots A = 0, \text{ or}$$

$$\pm 1 \times k^n - \dots n - 1 k^{n-1} \times u \dots \frac{n-1}{1} \times \frac{n-1}{2} k^{n-2} \times u u, \&c.$$

$$\pm a \times k^{n-1} - \dots n - 1 k^{n-2} \times u \dots \frac{n-1}{1} \times \frac{n-2}{2} k^{n-3} \times u u, \&c.$$

$$\pm b \times k^{n-2} - \dots n - 2 k^{n-3} \times u \dots \frac{n-2}{1} \times \frac{n-3}{2} k^{n-4} \times u u, \&c.$$

$$\pm c \times k^{n-3} - \dots n - 3 k^{n-4} \times u \dots \frac{n-3}{1} \times \frac{n-4}{2} k^{n-5} \times u u, \&c.$$

$\pm \&c.$

$\pm A = 0$

$\pm$

$\pm$

$\pm$

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I. Now by rejecting the Powers of u , and their Coefficients, it may be express'd by a Simple Equation, and its True value found by repeating the Operation so far as is necessary.

For since u is greater than it should be when it has a Positive, but less when it has a Negative sign; therefore 'tis plain, that if the supposed Root be less than the required Root, the following Operation will make it greater, and if it be greater, 'twill continue so, tho' less than the former, therefore the Root must necessarily Converge; and after an Infinite Convergence (if need be) must become equal to that sought: And therefore also the rest of the of the Terms may be safely rejected.

Theref.

Therof, $u = \frac{A - 1k^n + ak^{n-1} - bk^{n-2} + ck^{n-3}, \&c.}{nk^{n-1} + n-1ak^{n-2} + n-2bk^{n-3} + n-3ck^{n-4}}$

And since $k + u = x$, by *Supposition*, therefore

$x = \frac{A + n-1k^n + n-2ak^{n-1} + n-3bk^{n-2} + n-4ck^{n-3}, \&c.}{nk^{n-1} + n-1ak^{n-2} + n-2bk^{n-3} + n-3ck^{n-4}, \&c.}$

Whence *Particular Theorems*, (agreeing with those gi- by the Learned and Ingenious Mr. *Raphson*, in his *Analysis Equationum*) are easily drawn, for Extracting the Roots of all sorts of *Equations*, however *Compounded* or *Affected*.

And whatever *Term* is wanting in the *Equation*, must be omitted in the *Theorem*.

As 1. For all *Pure Powers*, i. e. if $x^n = A$,

Then $u = \frac{A - k^n}{nk^{n-1}}$; and $x = \frac{A + n-1k^n}{nk^{n-1}}$

2. If $+xx + ax = A$, then,

$u = \frac{A - k^2 + ak}{2k + a}$; and $x = \frac{A + k^2}{2k + a}$

And the like in any other *Equation*.

NOTE, That k may be taken at Pleasure; then in renewing the *Operation*, if you use the

<i>First Theorem, Let</i>				<i>Second Theorem, Let</i>			
1ft.	{	1ft.	{	2d.	{	1ft.	{
2d.		2d.		2d.		2d.	
3d.		3d.		3d.		3d.	
&c.		&c.		&c.		&c.	

$k \pm u = k$; $x = k$

II. Since $x = k \pm u$, by *Supposition*; Therefore

$x^n = k^n \pm \frac{n}{1}k^{n-1}u \pm \frac{n}{1} \times \frac{n-1}{2}k^{n-2}u^2 \pm \&c.$

And

And if x^n be $\begin{cases} \text{greater} \\ \text{Less} \end{cases}$ than k^n , 'twill be $k \pm u$:

$$\text{Let } (x^n \div k^n =) \frac{n}{1} k^{n-1} u \dots \frac{n}{1} \times \frac{n-1}{2} k^{n-2} u^2 \dots \&c. = m$$

Therefore u taken equal to

$$m \div \left\{ \begin{array}{l} \frac{n}{1} k^{n-1} (=a) \quad \text{Doubles} \\ \frac{n}{1} \times \frac{n-1}{2} k^{n-2} u (=b) \quad \text{Triples} \\ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} k^{n-3} u^2 (=c) \quad \text{Quadr.} \\ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} k^{n-4} u^3 (=d) \quad \text{Quint.} \\ \&c. \end{array} \right.$$

the *True Figures* in the assumed k , at each Operation.

$$\text{Since, } u = \frac{m}{n k^{n-1}}, \text{ or } \frac{m}{n k^{n-1} \pm \frac{1}{2} n n - n k^{n-2} u}$$

Therefore 'tis equal to

$$\frac{m}{n k^{n-1} \pm \frac{1}{2} n^2 - n k^{n-2} \times \frac{m}{n k^{n-1}}} \text{ or } \frac{m}{n k^{n-1} \pm \frac{n-1}{2k} m}$$

That is $u = \frac{m k}{n k^n \pm \frac{1}{2} n - 1 m}$, which is the *Rational*

Theorem given by Mr. Halley, (in *Philos. Trans.* N. 210.)
for *Extravelling the Roots of all Pure Powers.*

Also

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Alſo, $\pm nk^{n-1}u + \frac{1}{2}nn - nk^{n-2}uu, &c. = m$, by *Hyp.*

Then $uu \pm \frac{2k}{n-1}u = \frac{m}{\frac{1}{2}nn - nk^{n-2}}$ by *Equal Diviſion.*

And $u = \pm \frac{k}{n-1} + \frac{k}{n-1} \left| \frac{\frac{1}{2}m}{\frac{1}{2}nn - nk^{n-2}} \right|^{\frac{1}{2}}$, Theref.

$(k^n \pm m)^{\frac{1}{n}}$ or $k \pm u = \frac{n-2}{n-1}k + \frac{1}{n-1}k^2 + \frac{m}{\frac{1}{2}nn - nk^{n-2}} \left| \frac{1}{2} \right|^{\frac{1}{2}}$

Which is the *Irrational Theorem* given for the ſame purpoſe.

Note, If the *Root* of a very *high Power* be required; the renewing the *Theorem* will be ſomewhat troubleſome: Therefore having found 3 or 4 *Figures* of *u*, the reſt may be attain'd much eaſier, by applying the following *Correction*, viz. if it be $k \pm u$.

$$\frac{1}{2}n-2 \times \frac{1}{k} + \frac{1}{2}n-2 \times \frac{1}{4}n-3 \times \frac{1}{k^2} + \frac{1}{2}n-2 \times \frac{1}{4}n-3 \times \frac{1}{7}n-4 \times \frac{1}{k^3} &c.$$

$$\frac{1}{n-1}k^2 + \frac{1}{\frac{1}{2}nn - nk^{n-2}}m \left| \frac{1}{2} \right| \times 2$$

In repeating the *Correction*, the *u* laſt found muſt be uſed; and in the 1^{ſt}. 2^d. 3^d. &c. *Correction*, the *Denominator* muſt have 2, 4, 6, &c. *Terms*.

Alſo the *Diviſor* muſt be corrected every *Operation*, by *ſubſtracting*, or *Adding* the laſt *Correction*, from, or to it, if it be $k \pm u$.

Thus, if $x^{16} = 1,06$, 'tis found, with wonderful facility, by only one *Suppoſition* of $x = 1$, and *Corrections*, that $x = 1,008596535874329474417154$, &c. which would have bin an intolerable Labour to perform by any other Method.

Cc

III. Whence

III. Whence also may be had variety of other *Theorems* for finding the *Roots* of *Equations*: By resuming the former *General Equation*, where

$$A = p \dots q u \dots r u^2 \dots s u^3 \dots t u^4 \dots \&c.$$

And for *Equations* under the 9th *Dimension*, the following *Table* will abundantly assist the *Practitioner*; where

$$x^9 \dots a x^8 \dots b x^7 \dots c x^6 \dots d x^5 \dots e x^4 \dots f x^3 \dots g x^2 \dots h x \dots A = 0.$$

	9	8	7	6	5	4	3	2	1	
p	k^9	$a k^8$	$b k^7$	$c k^6$	$d k^5$	$e k^4$	$f k^3$	$g k^2$	$h k$	A
q	$9 k^8$	$8 a k^7$	$7 b k^6$	$6 c k^5$	$5 d k^4$	$4 e k^3$	$3 f k^2$	$2 g k$	b	$x u$
r	$36 k^7$	$28 a k^6$	$21 b k^5$	$15 c k^4$	$10 d k^3$	$6 e k^2$	$3 f k$	g	$x u^2$	
s	$84 k^6$	$56 a k^5$	$35 b k^4$	$20 c k^3$	$10 d k^2$	$4 e k$	f	$x u^3$		
t	$126 k^5$	$70 a k^4$	$35 b k^3$	$15 c k^2$	$5 d k$	e	$x u^4$			
v	$126 k^4$	$56 a k^3$	$21 b k^2$	$6 c k$	d	$x u^5$				
w	$84 k^3$	$28 a k^2$	$7 b k$	c	$x u^6$					
y	$36 k^2$	$8 a k$	b	$x u^7$						
z	$9 k$	a	$x u^8$							

1. Observe, That the *Terms* in the *Ranks* p, r, t, &c. must have the *same Signs* with their respective *ones* in the *Equation*.

2. If it be $k \pm p$, then $k \mp u = x$.

3. If it be $k \pm u$, then the *Terms* in the *Ranks* q, s, v, must have the *same Signs* with, or different from their respective *ones* in the given *Equation*.

And since $p \dots q u \dots r u^2 \dots s u^3 \dots t u^4 \dots \&c. \dots A = 0$.

Then $p = \dots q u \dots r u^2 \dots \&c.$ Therefore u taken = to

$\left. \begin{array}{l} q, \\ q, x u \\ q, r u, s u^2 \\ q, r u, s u^2, t u^3 \\ \&c. \end{array} \right\}$
 $\left. \begin{array}{l} \text{Doubles} \\ \text{Triples} \\ \text{Quadr.} \\ \text{Quint.} \\ \&c. \end{array} \right\}$
 The *True Figures* in the assumed k , at each *Operation*.

Since

$$\text{Since } u = \frac{p}{q} \text{ or } \frac{p}{q \pm ru} \text{ or } \frac{p}{q \pm rp} = \frac{qp}{qq \pm rp}$$

Therof. $x = k \pm \frac{qp}{qq \pm rp}$; which is the *Rational Theorem* for finding the *Roots of Affected Equations* of what *Power* soever.

Also because $qu \pm ru^2 = \pm p$ by *Supposition*.

Therefore, if p and A have *like*, or *unlike*; or p and r , *unlike*, or *like Signs*, 'twill be $k + u$, or $k - u$; and $u = \frac{\frac{1}{2}q \sqrt{\frac{1}{4}qq \pm pr}}{r}$; $\therefore x = k \pm \frac{\frac{1}{2}q \sqrt{\frac{1}{4}qq \pm pr}}{r}$

which is the *Irrational Theorem* for solving *Affected Equations*.

And by renewing the *Calculation* (if need be) and taking always the x last found for the k in the *New Equation*, 'tis easy to proceed to any exactness. But instead of renewing the *Theorem*, you may advantageously use the following *Correction*.

$$\text{viz. } \frac{su^3, tu^4, vu^5, wu^6, \&c.}{2 \times \frac{1}{4}qq \pm pr} \text{ or } \frac{\frac{1}{2}su^3, \frac{1}{2}tu^4, \frac{1}{2}vu^5 \&c.}{\frac{1}{4}qq \pm pr}$$

according as q and r have *like*, or *unlike Signs*. Where, in the *Dividend*, those *Terms* of the *Equation* that have the *same Signs* with, or different from s , must be $+$, or $-$.

And, in the *Divisor*, $\frac{1}{4}qq \pm pr$ contrary to, or the *same* with the *Sign* s , if it be $k - u$, but on the contrary, if it be $k + u$.

Also, in repeating the *Correction*, the *Divisor* may be corrected by *Subducting*, or *Adding* the last *Correction* multiplied by r , from, or to u , if s be *Affirmative*, or *Negative*;

Cc 2

gative ;

Positive; Then if q and r have the same, or different Signs, the new Correction must have the same Signs with, or different Signs from s .

E X A M P L E.

Suppose the Value of x in this Cubic Equation

$$x^3 + 438x^2 - 7825x - 98508430 = 0.$$

$$\text{or } xxx + axx - bx - A = 0. \text{ was required.}$$

Let $x = k \pm u$; (which k should be assum'd as near as possible to x , because convenient, but not necessary); therefore,

1. Suppose $k = 300$, then

$$\begin{array}{rcl} x^3 & = & +27000000 + 270000u + 900u^2 + u^3 \\ + ax^2 & = & +39420000 + 262800u + 438u^2 \\ - bx & = & -2347500 - 7825u \\ - A & = & -98508430 \end{array} \quad \left. \vphantom{\begin{array}{rcl} x^3 \\ + ax^2 \\ - bx \\ - A \end{array}} \right\} = 0$$

$$\text{i.e. } \left\{ \begin{array}{l} -34435930 + 524975u + 1338u^2 = 0 \\ -p + qu + ruu = 0 \end{array} \right\} \therefore x = k + u$$

Then by the Rational Theorem

$$u (= \frac{p}{q + \frac{rp}{q}}) = 56, 2; \text{ theref. } x (= k + u) = 356, 2$$

And by the Irrational Theorem

$$u (= \frac{\sqrt{qq + rp} - q}{r}) = 57; \text{ theref. } x (= k + u) = 357 +$$

2. Renew the Operation; And let $k = 356$, then,

$$\begin{array}{rcl} x^3 & = & +45118016 + 380208u + 1068u^2 + u^3 \\ + ax^2 & = & +55510368 + 311856u + 438u^2 \\ - bx & = & -2785700 - 7825u \\ - A & = & -98508430 \end{array} \quad \left. \vphantom{\begin{array}{rcl} x^3 \\ + ax^2 \\ - bx \\ - A \end{array}} \right\} = 0$$

$$\text{i.e. } \left\{ \begin{array}{l} -665746 + 684239u + 1506u^2 = 0 \\ -p + qu + ruu = 0 \end{array} \right\} \therefore x = k + u$$

Then

Then by the *Rational Theorem*

$$u \left(= \frac{p}{q + \frac{rp}{q}} \right) =, 970894 \therefore x = 356, 970894 -.$$

And by the *Irrational Theorem*

$$u \left(= \frac{\frac{1}{2}qq + rp \sqrt{\frac{1}{2}} - q}{r} \right) =, 970898 \therefore x = 356, 970898 +$$

Or supposing $k = 357$, Then,

$$\left. \begin{array}{r} x^3 = +45499293 - 382347u + 1071uu \\ + ax^2 = +55822662 - 312732u + 438uu \\ - bx = -2793525 + 7825u \\ - A = -98508430 \end{array} \right\} = 0.$$

$$i.e. \left\{ \begin{array}{l} +20000 - 687254u + 1509uu = 0 \\ + p - qu + ruu = 0 \end{array} \right\} \therefore x = k - u$$

Then by the *Rational Theorem*

$$u \left(= \frac{p}{q - \frac{rp}{q}} \right) =, 02910318169 \therefore x = 356, 9708968183 +$$

And by the *Irrational Theorem*

$$u \left(= \frac{\frac{1}{2}q - \frac{1}{2}q - rp \sqrt{\frac{1}{2}}}{r} \right) =, 02910318180, \text{ therefore,} \\ x = 356, 9708968182 -$$

If more Accuracy were required, repeating the *Operation*, or applying the *Correction*, would give any desired Number of *Figures* true in the *Root*.

The vast Advantage of this *Method*, beyond any other whatever, will be evident to them that judiciously examine each; who will also find it, considering the several Compendiously easy ways of *Operation*, together with

with its Swiftneſs of *Converging*, ſo Compleat, that no *Useful Improvement* therein can be well expected.

C H A P. IX.

Of Geometric Progreſſion.

D E F I N I T I O N.

A Continued Geometric Proportion, *that is, where the Terms do Increate or Decrease by Equal Ratio's, is called a Geometric Progreſſion; thus,*

$$a, ar, ar^2, ar^3, \text{ \&c. } \text{Incr.} \left. \begin{array}{l} \text{from a Con-} \\ \text{tinual} \end{array} \right\} \begin{array}{l} \text{Mult.} \\ \text{Diviſ.} \end{array} \left. \begin{array}{l} \\ \end{array} \right\} \text{by } r.$$

$$a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \text{ \&c. } \text{Decr.}$$

S C H O L I U M I.

But ſince this *Progreſſion* is only a *Compound* of two *Series*, viz.

$$\text{Of } \left\{ \begin{array}{l} \text{Equals} \\ \text{Geom. Proport.} \end{array} \right. \begin{array}{l} a, a, a, a, a, a, \end{array} \left. \begin{array}{l} \\ \end{array} \right\} \text{ \&c.}$$

$$1, r, r^2, r^3, r^4, r^5, \left. \begin{array}{l} \\ \end{array} \right\} \text{ \&c.}$$

Therefore, the moſt Natural *Progreſſion* is that which begins, with 1.

$$\text{As } \frac{1}{1}, \frac{r}{1}, \frac{r^2}{1}, \frac{r^3}{1}, \frac{r^4}{1}, \frac{r^5}{1} \left. \begin{array}{l} \\ \end{array} \right\} \text{ \&c. } \text{Increasing.}$$

$$\text{i. e. } 1, r, r^2, r^3, r^4, r^5$$

$$\text{And } \frac{1}{1}, \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \frac{1}{r^4}, \frac{1}{r^5} \left. \begin{array}{l} \\ \end{array} \right\} \text{ \&c. } \text{Decreasing.}$$

$$\text{i. e. } 1, r^{-1}, r^{-2}, r^{-3}, r^{-4}, r^{-5}$$

S C H O-

SCHOLIUM 2.

In Geometric Progression ;

If $\left\{ \begin{matrix} a \\ r \\ n \\ l \\ s \end{matrix} \right\}$ be the $\left\{ \begin{matrix} \text{First Term.} \\ \text{Ratio.} \\ \text{Number of Terms.} \\ \text{Last Term.} \\ \text{Sum of all the Terms.} \end{matrix} \right\}$

Then any *three* of these *Terms* being *given*, the other *two* are easily found.

And the several *Cases* are reducible to *Ten Propositions*, which are all Solved by the *Two* following *Lemmata*.

I. Of *Increasing Progressions*.

LEMMA 1.

In an *Increasing Geometric Progression*.

$a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, \&c.$

'Tis $1 : r :: s-1 : s-a$.

For $a : ar :: 1-1 : s-a$, } by *Cor.* { 13 } Th. 2.

But $a : ar :: 1 : r$ }
Therfore $1 : r :: s-1 : s-a$, }

COROLLARIES.

$$1. s = \frac{rl-a}{r-1} = \frac{l-a}{r-1} + l.$$

$$2. r = \frac{s-a}{s-l} = \frac{s-a}{s-l} \times \frac{l}{s-l}.$$

$$3. a = s + rl - rs = rl - s \times r - 1.$$

$$4. l = \frac{rs-s+a}{r} = \frac{a+r-1 \times s}{r} = s - \frac{s-a}{r}.$$

LEM.

L E M M A. 2.

In an *Increasing Geometric Progression.*

$$\text{'Tis } 1 : r^{n-1} :: a : l.$$

For $a, ar, ar^2, ar^3, ar^4, \&c., ar^{n-1} = l$

$$\text{Theref. } 1 : r^{n-1} :: a : l.$$

C O L L A R I E S.

$$1. l = ar^{n-1} = a \times r^{n-1}$$

$$2. a = \frac{l}{r^{n-1}} = l \times \frac{1}{r^{n-1}}$$

$$3. n = \left(\frac{L, r^{n-1}}{L, r} + 1 \right) = \frac{L, l - L, a}{L, r} + 1$$

$$4. r = \sqrt[n-1]{l \div a}$$

P R O P O S I T I O N S.

I. Given a, r, n ; Required l, s .

$$1. l = ar^{n-1} (= a \times r^{n-1}) \text{ by Lem. 2.}$$

$$\text{But } s = \frac{rl - a}{r - 1} \text{ by Lem. 1. And } r \times l = ar^n \text{ by Mult.}$$

$$2. \text{Theref. } s = \frac{ar^n - a}{r - 1} (= a \times \frac{r^n - 1}{r - 1}) \text{ by Substitution.}$$

II. Given a, r, l ; Required s, n .

$$1. s = \frac{rl - a}{r - 1} (= \frac{l - a}{r - 1} + l) \text{ by Lem. 1.}$$

$$2. n = \frac{L, l - L, a}{L, r} + 1 \text{ by Lem. 2.}$$

P R O.

III. Given a, r, s ; Required l, n .

1.
$$= \frac{\overline{r-1} \times s + a}{r} (= ar^{n-1}) \text{ by Lem. 1st. and 2d.}$$

Then $\overline{r-1} \times s + a (= r \times ar^{n-1}) = ar^n \text{ by Mult.}$

And $r^n = \frac{\overline{r-1} \times s + a}{a} \text{ by Divif. But } L, r = L, \frac{\overline{r-1} \times s + a}{a}$

2. Therefore $n = \frac{L, \overline{r-1} \times s + a - L, a}{L, r} \text{ by Division.}$

IV. Given a, l, s ; Required r, n .

1. $r = \frac{s-a}{s-l} (= \overline{s-a} \times \frac{1}{s-l}) \text{ by Lem. 1.}$

2. And $n (= \frac{L, l - L, a}{L, r} + 1) = \frac{L, l - L, a}{L, s - a - L, s - l} + 1 \text{ (by L. 2.)}$

V. Given a, n, s ; Required r, l .

Since $\frac{sr - s + a}{r} (= l) = ar^{n-1} \text{ by Lem. 1st. and 2d.}$

Then $sr - ar^n = s - a \text{ by Divif. and Transp.}$

1. Therefore $-r^n + \frac{s}{a}r = \frac{s-a}{a} = \frac{s}{a} - 1, \text{ by Divif.}$

And since $l = ar^{n-1}$, and $r = \frac{s-a}{s-l}$, therefore $l = a \times \frac{s-a}{s-l}^{n-1}$

2. Therefore $l \times \overline{s-l}^{n-1} = a \times \overline{s-a}^{n-1} \text{ by Mult.}$

D d

VI. Given

VI. Given a, n, l ; Required r, s .

1. $r = \frac{l-a}{l-a} \frac{1}{r^{n-1}}$ by Lem. 2. But $\frac{l-a}{r-1} + l = s$ by L. 1.
2. Theref. $s = \frac{l-a}{\frac{l-a}{r^{n-1}} - 1} + l$, by Substitution.

VII. Given r, n, l ; Required a, s .

1. $a = \frac{l}{r^{n-1}}$ by Lem. 2. But $\frac{lr-a}{r-1} = s$, by Lem. 1.
2. Theref. $s = \frac{lr - \frac{l}{r^{n-1}}}{r-1} = \frac{lr^n - l}{r^n - r^{n-1}}$ by Substitution.

VIII. Given r, n, s ; Required a, l .

Since $sr - s + a = ar^n$ by Lem. 1st. and 2d.

Then $sr - s = ar^n - a (= a \times r^n - 1)$ by Transp.

1. Theref. $a = \frac{sr-s}{r^n-1} = \frac{r-1}{r^n-1} \times s$ by Division.

And since $s = \frac{lr^n - l}{r^n - r^{n-1}}$ by Prop. 7. $\therefore sr^n - sr^{n-1} = lr^n - l$;

2. Theref. $l = \frac{sr^n - sr^{n-1}}{r^n - 1}$ by Division.

IX. Given r, l, s ; Required a, n .

1. $a = s + rl - rs (= lr - s \times r - 1)$ by Lem. 1.

But $\frac{l}{a} (= \frac{l}{s + rl - rs}) = r^{n-1}$ by Lem. 2.

And

And $L, \frac{l}{s+r-l-rs} = \overline{s-1} L, r$ by the Nat. of Log.

2. Theref. $n = \frac{L, l - L, s+r-l-rs}{L, r} + 1$, by Div. and Tr.

X. Given n, l, s ; Required r, a .

Since $sr^n - sr^{n-1} = lr^n - l$, by Prop. 8.

Then $l = sr^{n-1} - sr^n + lr^n (= sr^{n-1} - s + l \times r^n)$ by Transf.

1. Theref. $-r^n + \frac{s}{s-l} r^{n-1} (= \frac{s}{s-l} r^{n-1} - r^n) = \frac{l}{s-l}$

2. $a \times s - a^{n-1} = l \times s - l^{n-1}$ by Prop. 5.

II. Of Decreasing Geometric Progressions.

In *Finite Decreasing Progressions*, the same *Rules* will serve for the like *Propositions*, if the *Series* be inverted, so that the *Least Term* be the *First*, and the *Greatest* the *Last*.

And since in the *Increasing Geometric Progression*

$a, ar, ar^2, ar^3, ar^4, ar^5$, &c. to $ar^{n-1} = l$.

'Tis $r - 1 : 1 :: l - a : s - l$.

Therefore in a *Decreasing Geometric Progression*

'Tis $r - 1 : 1 :: a - l : s - a$. by Inverting the Terms

C O R O L L A R I E S.

1. But in an *Infinite Decreasing Progression* $l = 0$;
Therefore $r - 1 : 1 :: a : s - a$. whence,

D d 2

Given

Prop.	Given	Requd.	Solution
1.	a .	s .	$r a \div r - 1$
2.	s, a .	r .	$s \div s - a$
4.	s, r .	a .	$s - s \div r$

2. Also $a - \frac{a}{r} : \frac{a}{r}$ (i.e. 1st. — 2d. : 2d. or x) :: $r - 1 : 1$.

Therof. $a - x : x :: \left\{ \frac{a-1}{a} \right\} : s-a$, in $\left\{ \begin{array}{l} \text{a Finite} \\ \text{an Infinite} \end{array} \right\}$ Progr.

And $s = \frac{a a - x l}{a - x}$ in a *Finite*, or $s = \frac{a a}{a - x}$ in an *Infinite Decreasing Progression*.

Question. Suppose a Body should move at this rate, viz. in the 1st. Moment 10 Miles, in the 2d. 9 Miles, in the 3d. $8\frac{1}{2}$ &c. eternally, as 10 to 9;

Here is given $r = \frac{10}{9}$, $a = 10$; required s ; then,

By Cor. $\sum_2^1 \left\{ s \right\} = \left\{ \frac{r a \div r - 1}{a a \div a - x} \right\} = 100$ Miles fought.

That is, a *Moveable Body* continuing its Motion in that Ratio eternally, would only run 100 Miles, or more than any thing that is less than 100 Miles.

3. Since $r - 1 : 1 :: a : s - a$, therefore,

$$s - a = \frac{a}{r - 1} = a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3}, \text{ &c. } - a = \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3}, \text{ &c.}$$

Whence, if any Quantity a be continually divided by any other Quantity r , the Sum of all the Terms will be $\frac{a}{r - 1}$

i.e.

i. e. $\frac{1}{r-1} \times a$ or the $\frac{1}{r-1}$ of a .

Theref. $a \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}, \&c. = a \times \frac{1}{r-1}$

Or, $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}, \&c. = \frac{1}{r-1}$

Where, if $a = r - 1$, then $s = 1$.

4. Whence, 'tis evident, That an *Infinite Progression*, or an *Infinately Infinite one*, may be Collected into one *Sum*; which *Sum* may not be only *Finite*, but equal to *Nothing*.

And of *Infinites* 'tis hence plain, that some are *equal*, others *unequal*; and also that one *Infinite* may be equal to *Two* or more *Finites*, or *Infinites*.

C H A P. X.

Of Interest.

I. Of a Single Sum of Money Paid either Before, or After 'tis Due.

1. Allowing SIMPLE INTEREST:

Put $p =$ *Principal*, or *Sum forborn*.

$n =$ *Number of Years*, or *parts of a Year*.

$r =$ *Rate of* 11. *per Annum*.

$m =$ *Amount* of the said *Principal*, for that *Time*, at that *Rate*.

Since

Since the *Amount* of 1*l.* for 1 Year, is $1 + r$;

Therefore the *Amount* of 1*l.* for n Years is $1 + nr$;

And $1*l.* : 1 + nr :: p : p + pr = m$, the *Amount*,
and $prn = \text{Interest of the Principal } p$, at the *Rate* r , in
 n Years. Hence,

Probl.	Given	Reqd.	Given Solution
1.	$p, r, n.$	$m.$	$p \times 1 + r n$
2.	$m, r, n.$	$p.$	$m \div 1 + r n$
3.	$m, p, r.$	$n.$	$m - p \div p r$
4.	$m, p, n,$	$r.$	$m - p \div p n$

Whence variety of easy *Rules* for *Practice* may be deduced; such as, To find the *Interest* of any *Sum*, at any *Rate*, for any *Number* of days.

'Tis plain, that n Days $\times p$ Pence $\div 7300$ is the *Interest* in Pence, at 5*l.* per Cent. per An. Therefore,

$\frac{2r}{10} \times \frac{np}{7300}$, or $2r \times \frac{np}{73} \div 1000$, or $2r \times \frac{np}{7300} \div 10$. is the *Interest* in Pence, at the *Rate* r .

2. Allowing COMPOUND INTEREST.

Put $x (= 1 + r) = \text{Principal and Interest of 1*l.* for any given Time, at any given Rate. Then,$

Since, $1*l.* : x :: p : px = 1*st.*$
 $1*l.* : x :: px : px^2 = 2*d.*$
 $1*l.* : x :: px^2 : px^3 = 3*d.*$
 &c. &c. } Year Amount (m).

Therefore 'tis

$px^n = m$; whence,

Prob.

Probl.	Given	Reqd.	Solution
1.	$p, n, x.$	$m.$	$p \times x^n$ or $L, p + \overline{L, x \times n}$
2.	$m, n, x.$	$p.$	$m \div x^n$ or $L, m - \overline{L, x \times n}$
3.	$p, m, x.$	$n.$	$x^n \div x$ or $L, x^n \div L, x$
4.	$p, m, n.$	$x.$	$m \div p^n$ or $L, m - \overline{L, p \div n}$

Quest. 1. Suppose the Principal p were at Interest for n Times at the Rate r ; what is the Amount m ?

Quest. 2. If the Sum m was to be paid n Times hence; what is its present Value p discounting at the Rate r ?

Quest. 3. If the Principal p at the Rate r gives the Amount m ; what is the Time of forbearance n ?

Quest. 4. If the Principal p forborn n Times gives the Amount m ; what is the Rate r of Interest?

II. Of several Equal Payments at several Equal Times.

First, When paid after they are due;

Or Rules for finding the Amount, &c. of Annuities, Pensions, or Rents, &c. in Arrear,

1. Allowing SIMPLE INTEREST.

Put a for the Annuity, Rent, or Pension; then,

'Tis evident, $1l. : r :: a1. : ar$, that is,

The Interest of $a1.$ at the Rate r , per $1l.$ per An. is ar .

But $\left\{ \begin{array}{l} a + 1ar \\ a + 2ar \\ a + n-1 \times ar \end{array} \right\}$ is the $\left\{ \begin{array}{l} 1st. \\ 2d. \\ 3d. \\ nth. \end{array} \right\}$ Years Amount (m).

Therefore

Therefore $na + \frac{n(n-1)}{2} ar$ is the Sum of those Amounts, or the Amount (m) of the Annuity (a) at the n th. Year's end; whence,

Probl.	Given	Requd.	Solution
1.	$a, n, r;$	$m.$	$na + \frac{n(n-1)}{2} \times ar$
2.	$m, n, r;$	$a.$	$\frac{2m}{2 + nr - r \times n}$
3.	$m, n, a;$	$r.$	$\frac{m - na \times 2}{n - 1 \times na}$
4.	$m, r, a;$ Let $2a - ra = r$	$n.$	$\frac{r + \sqrt{r^2 + 8mra}}{2ra}$

2. ALLOWING COMPOUND INTEREST.

Since the Last Year's Annuity (a) carries out no Time, therefore no Interest can be demanded for it; consequently the First Year's Annuity will become ax^{n-1} ,

Wherefore, $a + ax + ax^2 + ax^3 + ax^4$, &c. to $ax^{n-1} = m$ the Amount

But $a : ax \} :: m - ax^{n-1} : m - a$, by Cor. 13. Th. 2.
That is $1 : x \}$

Therefore $mx - ax^n = m - a$, Whence,

Probl.

Probl.	Given	Reqd.	Solution
1.	$a, x, n.$	$m.$	$\frac{x^n - 1 \times a}{x - 1}$
2.	$m, x, n.$	$a.$	$\frac{x - 1 \times m}{x^n - 1}$
3.	$m, x, a.$	$n.$	$\frac{L, x - 1 \times m + a - L, a}{L, x}$
4.	$m, a, n.$	$x.$	$-x^n + \frac{m}{a} x = \frac{m - a}{a}$

Quest. 1. If the Annuity a be forborn n Times, at the rate r ; what Amount m will it arise to?

Quest. 2. If in n Times forbearance, at the Rate r the Amount m is raised; what was the Annuity a forborn?

Quest. 3. If the Annuity a , at the Rate r , raise or amount to m ; what was the Time n of forbearance?

Quest. 4. If the Annuity a forborn n Times, raise the Stock or Amount m ; what was the Rate r of Interest?

Secondly, *Being paid before they are due.*

Or Rules for finding the Discount, &c. in Buying and Selling of Annuities, Pensions and Leases in Reversions, &c.

I. Allowing SIMPLE INTEREST.

Since, As the Amount of 1 l. for any Time is to 1 l.
So is the Amount of an Annuity, to its Present value.

$$i. e. 1 + nr : 1 :: na + \frac{nn - n}{2} ar : \frac{na + \frac{1}{2}nn - nx ar}{1 + nr} = s.$$

E c. Therefore

Therefore,

Probl.	Given	Reqd.	Solution
1.	$a, r, n;$	$s.$	$\frac{2n + nnr - nr \times a}{2 + 2nr}$
2.	$s, r, n;$	$a.$	$\frac{2 + 2nr \times s}{2 + nr - r \times n}$
3.	$s, n, a;$	$r.$	$\frac{na - s \times 2}{2s - an + a \times n}$
4.	$s, r, a;$ Let $2sr + ra - 2a = r$	$n.$	$\frac{r + \sqrt{r^2 + 8sar}}{2ra}^{\frac{1}{2}}$

2. Allowing COMPOUND INTEREST.

$$\text{Since, } \left\{ \begin{array}{l} x : 1 l. :: a : \frac{a}{x} \\ x : 1 l. :: \frac{a}{x} : \frac{a}{x^2} \\ x : 1 l. :: \frac{a}{x^2} : \frac{a}{x^3} \\ \text{\&c.} \end{array} \right\} = \text{Present Value (s) at the } \left\{ \begin{array}{l} 1 \text{ ft.} \\ 2 \text{ d.} \\ 3 \text{ d.} \\ \text{\&c.} \end{array} \right\} \text{ Year's end.}$$

Theref. $\frac{a}{x} + \frac{a}{x^2} + \frac{a}{x^3} + \frac{a}{x^4}$, &c. to $\frac{a}{x^n} = \text{Present value } s.$

But $\frac{a}{x} : \frac{a}{x^2} :: s - \frac{a}{x^n} : s - \frac{a}{x}$ by Cor. 13. Ib. 2. Ch. 3.
or $x : 1$

Theref. $s - \frac{a}{x^n} = sx - a$: Whence follows,

Probl.

Probl.	Given	Reqd.	Solution
1.	$a, x, n.$	$s.$	$a - \frac{a}{x^n} \div x - 1$
2.	$s, x, n.$	$a.$	$\frac{x^n \times x - 1 \times s}{x^n - 1}$
3.	$s, x, a,$	$n.$	$\frac{L, a - L, a + s - sx}{L, x}$
4.	$s, a, n,$	$x.$	$-x^{n+1} + \frac{a+s}{s} x^n = \frac{a}{s}$

Quest. 1. *The Annuity a is to be sold for n Tears, allowing the Purchaser the Rate r; what is the Present worth s of the Annuity?*

Quest. 2. *Having the Sum s ready to be laid out, at the Rate r, to buy an Annuity for n Tears; what Annuity will it Purchase?*

Quest. 3. *The Annuity a is to be sold for n Tears, for the ready Sum s; what rate r has the Purchaser for his Money?*

Quest. 4. *The Annuity a is made over for Payment of a Debt s, allowing the Creditor the Rate of Interest r; In what Time n will the Debt be paid?*

Several other more Practical Rules for solving the last Problem in this, and the former Case, are easily found; we as shall shew hereafter.

C O R O L L A R Y 1.

By Supposing n , in the last Theorems, to be Infinite, and a the Annual Rent; it follows that $s = sx - a$.

Whence, Rules are drawn for Buying and Selling of Estates in Fee-Simple, allowing Com. Interest.

Prop.	Given	Reqd.	Solution
1.	$a, x.$	$s.$	$a \div x - 1$
2.	$s, x.$	$a.$	$s \times x - 1$
3.	$s, a.$	$x.$	$s + a \div s$

Quest. 1. There is a Fee-Simple to be sold, of a l. per Annum, allowing the Purchaser the Rate x , Comp. Interest; what Sum s will Purchase this Estate?

Quest. 2. There is a Sum s , ready to be laid out, at the Rate x , Comp. Interest, for Buying a Fee-Simple; what Yearly Rent can such an Estate be of?

Quest. 3. Having with the Sum s bought a Fee-Simple, whose Yearly Value is a l. what Rate x of Comp. Interest was allow'd for the Money?

COROLLARY 2.

Whence also, If it be required, how many Years Purchase, any Annuity is worth, allowing Comp. Intr.

Suppose N the Number of Years sought;

Now, that $Na = s$ is evident; and $a = sx - s$ by Freq.

Theref. $N (= \frac{s}{sx - s}) = \frac{1}{x - 1}$ by Division; hence,

Probl.	Given	Reqd.	Solution
1.	$x.$	$N.$	$1 \div x - 1$
2.	$N.$	$x.$	$N + 1 \div N$

Quest. 1. *There is a Fee-Simple to be sold; how many Tears Purchase N is it worth, allowing the Purchaser the Rate r Comp. Interest?*

Quest. 2. *There is a Fee-Simple bought, and N Tears Purchase given for it; what Rate r of Com. Interest is the Money valued at?*

CHAP. XI.

I. Of Combinations of Quantities:

DEFINITION.

The several Ways or Different Cases, of Taking or Leaving any Number of Quantities, out of any Number of Things exposed, without regarding their Order or Places, are called the Combinations of Quantities.

Let N be the Number of Things exposed.

n the Number of Quantities to be taken or left.

I. The Combinations of 2 Quantities, in these 6, viz. a, b, c, d, e, f, are 15.

For with $\left\{ \begin{array}{l} a \text{ --- } ab, ac, ad, ae, af; 5. \\ b \text{ --- } bc, bd, be, bf; 4. \\ c \text{ --- } cd, ce, cf; 3. \\ d \text{ --- } de, df; 2. \\ e \text{ --- } ef; 1. \end{array} \right.$

Therefore the Combinations of 2 in 6 is 15;
which is a *Figurate Number* of the 3d. Order, whose Side
is

is $N-1$, or $N-\overline{n-1}$; Because 'tis the *Aggregate* of a *Series* of the 2d. Order.

2. The *Combinations* of 3 *Quantities* in these 6, are 20, viz.

with a ,	with b ,	with c ,	with d .
$abc, abd, abc, abf; 4.$	$bcd, bce, bcf; 3.$	$cde, cdf; 2.$	$def; 1$
$acd, ace, acf; 3.$	$bde, bdf; 2.$	$cef; 1.$	$\frac{1}{1}$
$ade, adf; 2.$	$bef; 1.$		$\frac{1}{3}$
$acf; \frac{1}{10}$	$\frac{1}{6}$		

And $10 + 6 + 3 + 1 = 20$; which is a *Figurate Number* of the 4th Order, whose side is $N-2$, or $N-\overline{n-1}$; Because 'tis the *Aggregate* of a *Series* of the 3d Order: The same in any other.

Hence, $n+1$ expresses what Order of *Figurate Numbers* to take.

If $n = \left\{ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \text{\&c.} \end{matrix} \right\}$ then $\left\{ \begin{matrix} 1 = s, \text{ or Side of the Fig. Number.} \\ N-0 \\ N-1 \\ N-2 \\ \text{\&c.} \end{matrix} \right\} = s = N - \overline{n-1}.$

Consequently, $\frac{s+0}{1} \times \frac{s+1}{2} \times \frac{s+2}{3}, \text{\&c. to } \frac{s+\overline{n-1}}{n}$ shall be the *Figurate Number* required, or the Number of *Combinations* of n *Quantities* in N . by *Prob. 4. Cb. 8.*

SCHOLIUM.

Whence, the usual *Examples*, given by *Writers* on this Subject, are easily *Solv'd*; and the Reason of those *Solutions* made evident.

II. Of

II. Of Elections of Quantities.

The Sum of all the Combinations found by taking 0, 1, 2, 3, &c. Quantities out of any Number of Things expos'd, is called the Election of Quantities.

Thus in 6 Quantities, the Elections are 64.

The Combinations of $\left\{ \begin{smallmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{smallmatrix} \right\}$ in 6, is $\left\{ \begin{smallmatrix} 1 \\ 6 \\ 15 \\ 20 \\ 15 \\ 6 \\ 1 \end{smallmatrix} \right\}$

Therefore the Election of 6 Quantities is 64: And so in any other Number.

Hence 'tis evident, that the several Combinations in any Number, are only the *Uncia* of the several Terms of a Binomial Root, rais'd to the Power N; or the Members of the respective Powers of $1 + 1$ taken as a Binomial:

Therefore $\overline{1+1}^N$ or 2^N is the Election of N Quantities, or the Number of Varieties in taking or leaving, 0, 1, 2, 3, &c. of any Number of Things expos'd, without regarding their Order.

Note, If the taking of 0 be excluded; Then $2^N - 1$ will be Election of N Quantities.

But if the taking of 0 and 1 be excluded; Then $2^N - \overline{N+1}$ will be the Elections of N Quantities.

III of

III. Of Permutations of Quantities.

DEFINITION.

The several ways, that the Order of a certain Number of Different Quantities, may be varied or differently Placed, are called Alterations, Variations, or Permutations of Quantities.

As suppose the Quantities were a, b, c, d , &c. 'Tis plain, In 1 Quant. a . In 2 Quant. a, b . In 3 Quant. a, b, c , &c.

$$\{a\}$$

$$1 = 1 \text{ var.}$$

$$\{ab\}$$

$$1 \times 2 = 2 \text{ variat.}$$

$$\{abc\}$$

$$1 \times 2 \times 3 = 6 \text{ Var.}$$

Theref. $N \times N - 1 \times N - 2 \times N - 3$, &c. to $N - N$ is the Number of Alterations, or Changes of Order in N Quantities

1. Hence the Changes that any Number of Bells admit of, are readily found.

2. As also, how many ways, the Letters of a Name, or Word may be differently disposed of, by way of Anagram; As the word Roma admits of $4 \times 3 \times 2 \times 1 = 24$ Changes.

But if one, or more of the Letters, do occur more than once; Then the whole Number of Changes, must be divided by the Product of the Changes of the Repetition of those Letters; Thus,

$$a a b c \text{ admits of } \frac{5 \times 4 \times 3 \times 2 \times 1}{6 \times 2} = \frac{120}{12} = 10 \text{ Changes.}$$

IV. Com-

IV. Composition of Quantities.

DEFINITION.

The several Dispositions of any Number of Quantities placed according to all the different ways, they are capable of, by taking them either singly, or connecting the given Rank to each Term therein, as often as required, are call'd Composition of Quantities: Thus,

The Composition of 2 Quantities in these 3, ¹⁰*a, b, c,* is 9.

For with *a. b. c.*

<u>a</u>	<u>b</u>	<u>c</u>
<i>a a</i>	<i>a b</i>	<i>a c</i>
<i>b a</i>	<i>b b</i>	<i>b c</i>
<i>c a</i>	<i>c b</i>	<i>c c</i>

$$3 + 3 + 3 = 9$$

And the Composition of 3 Quantities in Three is 27.
or Generally N^n is the Composition of n Quantities in N .

COROLLARY.

Whence because $N^1, N^2, N^3, N^4, \&c.$ to N^n or N^N is a *Geometric Progression*, whose *First Term* = N = *Ratio*, and *Number of Terms* = n . Therefore

$$\text{its Sum } (N \times \frac{N^n - 1}{N - 1} = \frac{NN^n - N}{N - 1}) N^{n-1} + \frac{N^n - 1}{N - 1}$$

(by *Prop. 1. Ch. 9.*) will give the several *Dispositions* there may be of N Quantities, taking them one and one, two and two, &c. to N Quantities.

FF

SCH O-

S C H Q L I U M.

And since $24^{24} (N^N) = 1333735776850284124449$
 081472843776 is the Number of the several Compositions
 that 24 Letters make, by taking each time 24; therefore,

$(N^N - 1 + N^N - 1 \div N - 1 =) 1391724288887292$
 999425128493407200 , is the Number of all the words
 significant and insignificant, that can be made of 24
 Letters.

Variety of Instances might have bin given on the se-
 veral Heads of this First Part, but because the Applica-
 tion is endless, the Learner (in most Places) is left to
 his Liberty of taking what Examples he pleases: For we
 would use as much brevity as possible, in Order that o-
 ther useful things design'd for this Treatise, may come
 in: And 'tis hoped what has already bin laid down, if
 well regarded, is sufficient without any farther illustra-
 tion, to render the whole Art of Nammoning, whether
 Theoretical or Practical, intelligible to any reasonable Ca-
 pacity.

The Symbols used in the following Part, are only the
 Common ones, as $\angle$, for Angle; $\perp$, for Right Angle; $\perp$
 for Perpendicular; $\parallel$, for Parallel; ∞ , for Infinite: with
 other Abbreviations of words, which the Reader will easily
 understand.

S Y N O P.



SYNOPSIS
Palmariorum Matheseos.
 PART II.
 Containing the
 PRINCIPLES
 O F
 GEOMETRY.

1. **T**HAT Science by which we learn how to Compare Things extended the one with the other, or determine how the one is More Less, or as Much extended as another, is called *GEOMETRY*.

2. In Things extended we distinguish three *Dimensions*, viz. *Length*, *Breadth*, and *Thickness*.

The Quantity wherein we consider $\left\{ \begin{array}{l} \text{Length, Breadth \& Thickness} \\ \text{only Length, and Breadth} \\ \text{nothing but Length} \end{array} \right\}$ is call'd $\left\{ \begin{array}{l} \text{Solid.} \\ \text{Surface.} \\ \text{Line.} \end{array} \right.$

And that wherein we consider neither *Length, Breadth, or Thickness*, and therefore is conceiv'd to have no Parts, or whose Divisibility is not consider'd, is call'd a *Point*.

3. *Lines, Surfaces, and Solids* may be consider'd as generated by a continual Motion of *Points, Lines, and Surfaces*: And as these Motions may be different, so may there be different Sorts of *Lines, Surfaces, and Solids* described by them.

4. If a *Point* move towards another the nearest way, 'twill describe a *Right Line*, otherwise a *Curv'd* one: So a *Right Line* moyed on a *Right, or Curved Line*, will produce a *Plane, or Curved Surface*: The same may be conceived of *Solids*.

5. The Curve-line whose parts are equally distant from a Point in the same Plane, is call'd a *Circumference*; that Point the *Centre*; Right Lines drawn from the Centre to the Circumference are call'd *Radii*: Those drawn from one Point of the Circumference to the other are call'd *Chords*; and that Chord which passes thro' the Centre is call'd a *Diameter* and divides the Circumference equally in two. Also any *Circumference* may be conceived to be divided into 360 equal parts call'd *Degrees*, and each Degree into 60 equal parts, call'd *Minutes*, and each of those again subdivided into 60 *Seconds*, &c. Any part of the Circumference is call'd an *Arc*.

6. The Inclination of two Right Lines meeting in a Point, so as not to make one Right Line, is call'd a *Plane Angle*; which therefore may be conceived to be generated by the Rotation of Sides: And its *Measure* is an Arc described from the Angular Point as a Centre, and intercepted between the Lines which form it. An *Angle* is said to be *Equal*, to, *Greater*, or *Less* than another, according as the Arc which measures it contains, as *Many, More, or Fewer* of the equal parts into which that Circumference is supposed to be divided.

7. When

7. When a Right Line ſtands upon another, ſo as to make equal *Inclinations* or *Angles* on each Side thereof, that Line is called a *Perpendicular*; and is the neareſt Diſtance between a given Point and a given Line; thoſe equal *Angles* are called *Right Angles*; which therefore have each for their Meaſure $\frac{1}{4}$ of the *Circumference* or 90 *Deg.* Conſeq. $\frac{1}{2}$ the *Circumf.* = two *Right Angles*; and the whole *Circumf.* = 4 $\angle$ s. An Angle greater, or leſs than a *Right Angle* is called *Obtuſe*, or *Acute*.

8. Half the Chord of twice any *Arc* is called the *Right Sine* of that *Arc*: That part intercepted between the *Arc* and its *Right Sine*, is called the *Versed Sine* thereof. And the *Sine* of an Angle is the *Sine* of the *Arc* which meaſures that Angle; and is a *Perpendicular* from one end of the *Arc*, on a *Right Line* drawn from the Angular Point to the other.

9. A *Right Line* falling upon another makes two *Angles* equal to two *Right Angles*. For they are meaſured by $\frac{1}{2}$ the *Circumference*, which meaſures 2 $\angle$ s (by 7.) Conſequently all the $\angle$ s made at the ſame Point, on the ſame ſide of a *Right Line*, are = 2 $\angle$ s. Therefore, that is a *Right Line* on which another falling makes two $\angle$ s = to 2 $\angle$ s. And all the $\angle$ s that can be made about any Point are = 4 $\angle$ s.

10. Therefore the op. $\angle$ s. (a, c) of croſſing Lines are =; For $a \perp b = 2 \angle$ s $\Rightarrow b \perp c$ (by 9) therefore $a \parallel c$ (by Subduſion.)



11. A *Tangent* to any Point of the *Circumference* is *Perpendicular* to the *Radius* drawn to that Point. For the *Radius* is the neareſt Diſtance.

12. A *Right Line* drawn *Perpendicular* to the *Radius*, paſſing thro' one end of an *Arc*, and limited by a Line, called *Secant*, drawn from the Centre thro' the other end, is the *Tangent* of that *Arc*.

13. The *Radius* being ſuppoſed divided into any Number of = parts; the Quantity of the *Sines*, *Tangents*, &c. of all *Angles* are eſtimated according as they contain more or fewer of ſuch Parts. Therefore,

In

In the same or equal Circumferences, the Sines, Tangents, &c. of equal Arcs or Angles are respectively equal. Because they consist of an equal Number of Parts of the same or equal Radii.

In unequal Circumferences, the Sines, Tangents, &c. of equal $\angle$ s, or Similar Arcs are similar also. Because they contain the like Number of Parts, of their respective Radii.

14. The Difference of an Arc from $\frac{1}{2}$, or $\frac{1}{4}$ the Circumference is called the Supplement, or Complement of that Arc. An Arc and its Supplement, have the same Sine, Tangent, and Secant. And the Sine, Tangent, or Secant of the Complement, is called the Co-Sine, Co-Tangent, or Co-Secant of the Arc, whose Complement that is.

15. Lines which no where incline one to the other are called Parallel Lines.

16. And two Parallel right Lines (P, p) have the same Inclination to a third Line (L). For Lines parallel may be taken as one broad Line.

17. If Parallel Right Lines be cut by a Right Line;

1. All the op. $\angle$ s are equal; And the contrary. For $a = c$, $e = g$ (by 16,) but $a = b$, $e = d$, $c = f$, $g = h$, (by 10,) Therefore $a = b = c = d$; and $e = f = g = h$ by Substitution.

2. The two op. $\angle$ s, whether internal or external are $= 2 \angle$ s; and the contrary. For

$f + a = b + e = c + d + g = 2 \angle$ s (by 9); but $a = c$, $e = g$ (by 16). Therefore $f + c = b + g = a + b = e + d = 2 \angle$ s by Substitution.

18. And, all Chords drawn parallel to the Tangent, are bisected by the Diameter passing thro' the Point of Contact. Because they are perpendicular to it (by 12,) and their Extremities are equally distant from the Centre, and also from the Point of Contact; therefore the Parts of the Chords will be Sines of equal Arcs, and consequ. equal (by 13).

Therefore if a Chord be bisected by another at Right Angles, the bisecting one is a Diameter: And the contrary.

19. Whence

19. Whence a Circumference may be drawn thro' any three Points, not in a Right Line: And the Intersection of two Right Lines bisecting the Distances between those Points at L s is the Centre; Since Right Lines bisecting Chords at L s are Diameters. (by 18.)

20. A bounded Space is called a *Figure*: If bounded by Right, or Curv'd Lines; 'tis called a *Rectilinear*, or *Curvilinear Plane Figure*.

21. Those Rectilinear Plane Figures, whose L s are respectively equal, and the Sides about those L s directly proportional, are called *Similar Rectilinear Figures*.

22. Those Rectilinear Plane Figures that are bounded by 3, or 4 Right Lines, are called *Trilateral* or *Triangular*, *Quadrilateral* or *Quadrangular Plane Figures*.

23. *Triangular Figures* are considered with respect to their *Sides* or *Angles*.

1. Those that have 3 equal, 2 equal, or 3 unequal Sides are called *Equilateral*, *Isosceles*, or *Scalenous Triangles*.

2. Those that have 1 Right L , 1 Obtuse, or 3 Acute L s are called *Right*, *Obtuse*, or *Acute L d Δ s*.

24. *Quadrilateral Figures*, whose opposite Sides are parallel, are called *Parallelograms*. And,

Right L d. $\S$ all $\S$ Sides equal $\S$ Squares.
Pgrs. having $\S$ only op. $\S$ are call'd $\S$ Rectangles.

An Obliq. L d. $\S$ all $\S$ Sides equal $\S$ Rhombus.
Pgr. having $\S$ only op. $\S$ is call'd a $\S$ Rhomboides.

25. And a *Parallelogram* may be conceived to be generated by a Right moveable Line drawn uniformly into the Length of a Right Immoveable one: Therefore the Surface generated by those Lines shall contain so many little Planes equiangular with the whole, as there are Units in the Product of their Parts.

26. Therefore Pgrs. and consequently Δ s having one L equal, are as the Product of the Sides about that L . And if those Sides be reciprocally $::1$, the *Parallelograms*, or Δ s shall be equal; and the contrary.

27. Also

27. Also, if h and b be the Height and Breadth of a Rectangle; its Area will be $h b$, from its Genesis, i. e. 'twill contain so many Square Areas as there are Units in the Product of its Height and Breadth.

And since the Contents of all Surfaces are estimated by the Square of some known Length; therefore such Figures must be reduced to Rectangular ones, before they can be Measured.

28. A Quadrilateral Figure of unequal Sides is called a Trapezium. And all other Rectilinear Figures, are in general, called Polygons, and have particular Names from the Number of their Sides or Angles, as those of 5 Sides Pentagons, of 6, Hexagons, of 7, Heptagons, &c.

All Plane Figures having equal, or unequal Sides or Angles, are called Regular, or Irregular Figures.

29. A Right Line drawn from L to L in any Figure, is called a Diagonal, or by some a Diameter.

The Height of any Figure is the nearest Distance between its Top and Base: Therefore all Figures between the same Parallels have the same Height.

30. A Plane Figure whose Extremities are equally distant from a Point therein, is called a Circle: And may be conceived to be generated by the Revolution of a right Line, one end being fix'd as the Centre, the other describing the Circumference of the Circle. And the Space, less than a Semi-Circle, described by a partial Conversion is called the Sector of a Circle. A Part of a Circle cut off by a Right Line is called the Segment of a Circle.

31. Hence, all Circumferences, as also like Arcs, their Sines, Tangents, &c. are as their Radii.

32. The Genesis of Solids may be exhibited various ways, some respecting the Dimension both of their Solidity and Surface; others, that of the Surface only. As,

1. A Parallelogram being conceived to move uniformly the Length of an immoveable Right Line, shall generate a Solid called a Parallelepiped: Or if the Describing Surface be a Square moving the Length of, and perpendicular to, its Side, the Solid is called a Cube: Therefore a Rectangular Parallelepiped contains so many equal Cubes, as there are Units in the Product of the Square Areas
in

in the deſcribing Plane, by the parts, of like meaſure for Length, in the Length moved: So that if b be its Height, and b the Area of its Baſe, then $bb = \text{Solid Content}$; or if l, b be the Length, and Breadth of the Baſe, then $bb l = \text{Solidity}$.

If the deſcribing Plane be a Polygon, or Circle, the generated Solid is called a *Prism*; or *Cylinder*.

But if the deſcribing Polygonal, or Circular Plane, in moving be ſuppoſed to decreaſe uniformly, 'till it comes to a Point, the Solid generated is called a *Pyramid*, or *Cone*.

2. A Right Line moving uniformly, ſo that its ends may deſcribe the Periphery of two parallel, ſimilar, and equal Polygons, will generate the Surface of a *Prism*, whoſe Baſes are thoſe Polygons.

If the Baſes are *Prs.* Δ s, or Circles, the Surface generated will be that of a *Parallelepiped*, *Triangular Prism*, or *Cylinder*.

The right Line connecting the Centres of the Polygonal Baſes is called the *Axis*.

And a right Line fix'd by one end to a Point, and with the other deſcribing a Polygon, or Circle, will generate the Surface of a *Pyramid*, or *Cone*, whoſe Baſe is that Polygon, or Circle.

The right Line connecting the fix'd Point, and the Centre of the Baſe is alſo called the *Axis*.

And that Solid whoſe *Axe* is perpendicular, or oblique to its Baſe is ſaid to be *Right*, or *Scalenous*.

The Conſervation of a Semicircle round the Diameter will generate a Solid called a *Sphere*.

Solids contain'd under an equal Number of like Surfaces, are ſaid to be *Similar*.

33. And Quantities, as alſo their Ratio's, that continually tend to an Equality, and therefore that approach nearer the one to the other, than any Difference that can poſſibly be aſſign'd, do at laſt become equal. For, either they'll be at laſt equal, or there will be ſome Difference (d) nearer than which they cannot approach; But they do continually

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ally approach by *Supposition*; therefore there can at last be no difference, therefore they must be equal.

Hence, all *Curved Lines* may be considered as composed of an Infinite Number of Infinitely little right Lines: And any one of them Produced, only Touches the Curve, therefore is called a *Tangent* to that Point of the *Curve*.

And as all *Surfaces* may be consider'd as composed of an Infinite Number of Parallel, Right, or Curved *Lines*, or *Surfaces* of an infinitely small Breadth.

So also *Solids*, of an Infinite Number of Parallel, Plane, or Curved *Surfaces*, or *Solids* of an infinitely small Thickness, called the *Elementa of Figures*.

34. And by considering Quantities as generated by continual Motion, 'tis apparent, that in equal Spaces of Time, they will become greater, or less proportionally as the Celerity of the Motion by which they are so generated is greater or less: Hence the Celerity of the Motion is very properly called *Fluxion*, and the Quantity generated *Fluent*.

Now these *Fluxions* of Quantities are in the *First Ratio* of their *Nascent Augments*; and may be express'd by *Finite Quantities* proportional to them.

And as x (by moving uniformly) becomes $x + o$, x^n becomes $x + o|^n = x^n + nx^{n-1} \times o + \frac{1}{2}n(n-1)x^{n-2} \times o^2 + \mathcal{O}c$. But the

Augments o , & $nx^{n-1} \times o + \frac{1}{2}n(n-1)x^{n-2} \times o^2 + \mathcal{O}c$.

are as 1 and $nx^{n-1} + \frac{1}{2}n(n-1)x^{n-2} \times o + \mathcal{O}c$. i. e. as 1 & nx^{n-1} .

Th. F, $x : F, x^n :: 1 : nx^{n-1}$

The *Flux.* of the *Fluents* $x, \dot{x}, \ddot{x}, \ddot{\dot{x}}, \mathcal{O}c$. are denoted by $\ddot{x}, \ddot{\dot{x}}, \ddot{\ddot{x}}, \ddot{\ddot{\dot{x}}}, \mathcal{O}c$. which are the 1st. 2d. 3d. 4th, &c. *Flux.* of x .

Also, as $\ddot{x} = F, \dot{x}$, and $\dot{x} = F, x$; So we may consider

sider $x = F, x'$, and $x = F, x'', \&c.$ As also denote the Fluxion of

$$\overline{ax - x^2}^{\frac{1}{2}}, \overline{ax - x^2}^{\frac{1}{2}}, \overline{ax - x^2}^{\frac{1}{2}}, \frac{ax + x x'}{a - x}$$

$$\text{by } \overline{ax - x x'}^{\frac{1}{2}}, \overline{ax - x x'}^{\frac{1}{2}}, \overline{ax - x x'}^{\frac{1}{2}}, \frac{ax + x x'}{a - x}$$

Hence $1 : nx^{n-1} :: x : nx^n = F, x^n$, whe-
ther n be Affirmative or Negative, Integer or Fraction,
as for Instance.

$$F, x^{-n} \left(\frac{1}{x^n} \right) \text{ is } (-nx^{n-1}) = -nx \times \frac{1}{x^{n+1}} = -\frac{nx}{x^{n+1}}$$

$$\text{Th. } F, \frac{1}{x} (x^{-1}) \text{ is } (-x^{-2}) = -\frac{x}{x^2}$$

$$F, x^{-\frac{m}{n}} \left(\frac{1}{x^{\frac{m}{n}}} \right) \text{ is } -\frac{m}{n} x^{-\frac{m}{n}-1}$$

$$F, x^{\frac{1}{2}} \text{ is } \left(\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} x \times x^{-\frac{3}{2}} = \frac{1}{2} x \times \frac{x}{x^{\frac{3}{2}}} \right) = \frac{x}{2x^{\frac{3}{2}}}$$

35. In any given Equation, involving Fluents, to find the Fluxion.

Rule. Multiply every Term of the Equation separately by the several Indices of the Powers of the Fluents therein; and in every such Product, change one of the Roots of the Powers into its Fluxion, the Aggregate of all the Products, connected under their proper Signs, will be the Fluxion of the Equation sought.

This Rule is usually demonstrated after this manner;

$$\text{If } x^3 - xy^2 + a^2 z - b^4 = 0; \text{ Then its Fluxion is}$$

$$3x^2 \dot{x} - x \dot{y}^2 - 2xy \dot{y} + a^2 \dot{z} = 0: \text{ For supp. } 0 \text{ an}$$

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Inf-

Infinitely small Quantity; let $o\dot{x}$, $o\dot{y}$, $o\dot{z}$, represent the Instantaneous Increments of the Fluents x , y , z ; These after a Momentary Increment of Time, will become

$x + o\dot{x}$, $y + o\dot{y}$, $z + o\dot{z}$, which being substituted instead of x , y , z , in the given Equation, 'twill be x^3

$+ 3x^2o\dot{x} + 3xo^2\dot{x}^2 + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xoy\dot{y} - 2\dot{x}o^2y\dot{y} - xo^2y^2 - \dot{x}o^3y^2 + a^2z + a^2o\dot{z} - b^3 = 0$. From this Subtract the given Equation; Then divide the Remainder by o ; and reject all Terms multiplied by it, (as being infinitely little;) There refts $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + a^2\dot{z} = 0$

Also, if $x^3 - xy^2 + a^2\sqrt{ax - y^2} - b^3 = 0$, then its Flux. is $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + a^2\sqrt{ax - y^2} = 0$.

Or let $z = \sqrt{ax - y^2}$, th. $\dot{z} = \frac{a\dot{x} - 2y\dot{y}}{2z}$; and (by this) $2z\dot{z} = a\dot{x} - 2y\dot{y}$, th. $\dot{z} = \frac{a\dot{x} - 2y\dot{y}}{2z}$, i. e. $\frac{a\dot{x} - 2y\dot{y}}{2\sqrt{ax - y^2}} = \frac{1}{2\sqrt{ax - y^2}}$

Th. $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + \frac{a^3\dot{x} - 2a^2y\dot{y}}{2\sqrt{ax - y^2}} = 0$

And by repeating the Operation, the *Second, Third, &c.* Fluxions of Equations are found.

If

If $\dot{r}y^3 - \dot{r}^4 + a^4 = 0$, then by Operation

$$1. \dot{r}y^3 + 3\dot{r}\dot{y}y^2 - 4\dot{r}^4 = 0$$

$$2. \ddot{r}y^3 + 6\dot{r}\ddot{y}y^2 + 3\dot{r}\dot{y}\dot{y}^2 + 6\dot{r}\dot{y}^2y - 4\ddot{r}r^3$$

$$- 12\dot{r}^2\dot{r}^2 = 0$$

Hence, The Fluxion of Quantities multiplied, is the Sum of the Products of the Fluxion of each Factor, by the Product of the other Factors. Thus,

$$F, xy \text{ is } \dot{x}y + x\dot{y}; \text{ And } F, by \text{ is } (\dot{b}y + 0y) b\dot{y}$$

$$F, \sqrt{2rx - x^2}^{\frac{1}{2}} \text{ is } \left(\frac{2r\dot{x} - 2x\dot{x}}{2 \times 2rx - x^2}^{\frac{1}{2}} \right) = \frac{r\dot{x} - x\dot{x}}{2rx - x^2}^{\frac{1}{2}}$$

$$F, \sqrt{rx + x^2}^{\frac{1}{2}} \text{ is } \frac{\dot{r}x + r\dot{x} + 2x\dot{x}}{2 \times rx + x^2}^{\frac{1}{2}}; \text{ And}$$

$$F, \sqrt{ax - r\dot{r}}^{\frac{1}{2}} \text{ is } 3 \times \frac{a\dot{x} - r\dot{r}}{ax - r\dot{r}}^{\frac{1}{2}} \times a\dot{x} - 2\dot{r}\dot{r}$$

and so in others: Also in Fractions

$$F, \frac{x}{y} \left(x \times \frac{1}{y} \right) \text{ is } \left(\dot{x} \times \frac{1}{y} - \frac{\dot{y}}{y^2} \times x \right) \frac{x\dot{y} - x\dot{y}}{yy}$$

$$F, \frac{a}{x} \left(a \times \frac{1}{x} \right) \text{ is } \left(0 \times \frac{1}{x} - \frac{\dot{x}}{x^2} \times a \right) = - \frac{a\dot{x}}{x^2}$$

Th. the Flux. of any Fraction, (N being Numerat. and D Demoninat.) is $\frac{ND - \dot{N}D}{D^2}$.

If the Indices are Flowing Quantities; as for instance,

$$F, y^{x+x} \text{ is } y^{x+x-1} + xyy^{x+x-1} = y^{x+x-1}, \text{ For}$$

$$y^x + F, y^x \left(= y + y \right)^{x+x} = y^{x+x} + xyy^{x+x-1}$$

Therefore if the Index be the Sum, Product, or Power, of Fluents; 'tis but substituting it, and its Flux. instead of x and $\dot{x}$ in this.

26. And

36. And the Fluent (ϕ) of $x^{\frac{r}{n}}$ is $\frac{x^{\frac{r}{n}+1}}{\frac{r}{n}+1}$

The Fluent of $a \dot{z}^r \dot{z} \times b \overline{+ c z^n}^m$ is easily found to be

$$\begin{aligned} & \frac{1}{m n + r + 1} \times \frac{a}{c} (A) \times \dot{z}^{r-n+1} \\ & + A \times \frac{n-r-1}{m n + r-n+1} \times \frac{a b}{c} (B) \times \dot{z}^{r-2n+1} \\ & + A B \times \frac{2n-r-1}{m n + r-2n+1} \times \frac{a b}{c} (C) \times \dot{z}^{r-3n+1} \\ & + \&c. \times \overline{b + c z^n}^{m+1} \end{aligned}$$

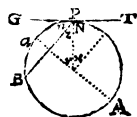
Hence ϕ , of $a \dot{y} + \overline{a y - y^2}^{\frac{1}{2}}$ is $\frac{2}{3} \overline{a y - y^2}^{\frac{3}{2}}$
And the like of others.

37. Where the *Extreme Value*, whether *Greatest* or *Least*, of any Quantity is required; since it is an Invariable by Sup. and its Flux. = 0; Therefore to determine it to an *Extremum*, Put the Equation into Fluxions, let the Fluxion of that Quantity = 0; then will all the Terms wherein 'tis found vanish, and the Extremum be determined by the remaining ones.

38. The $L(n, N)$ made by the Tangent (PG, PT) and Chord (PB) has for its measure an Arc = half that subtended by the Chord.

Draw a Diameter $\perp$ PB, then is the Arc a , and Chord PB bisected (by 18), draw the Rad. CP, as also $CD \parallel PB$.

Then $L n + L \dot{z} = L = L r + L x$
(by Constr.) But $\dot{z} = x$ (by 17) th. $n = r$ by Eq. Subduſt. And $\frac{1}{2} a$ Measures the $L r$ (by 6) therefore alſo the $L n$: Alſo $n + N$ are measured by $\frac{1}{2} a + \frac{1}{2} A$ (by 9) th. N is measured by $\frac{1}{2} A$.



39. An $L(c)$ in the Circumference made by two Chords, is measured by $\frac{1}{2}$ its subtending Arc A .

For

For $\frac{1}{2}N + A + M$ measures $n + c + m$ (by 9) But $\frac{1}{2}N, \frac{1}{2}M$ measures n, m (by 38) Therefore $\frac{1}{2}A$ measures c .



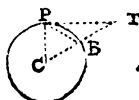
40. All $\angle$ s in the Circumference, subtended by the same Arc or Chord, are equal. For each is measured by $\frac{1}{2}$ the subtending Arc.

41. An $\angle$ (C) at the Centre is double that (c) at the Circumference standing on the same Arc (A.) For C is measured by the whole (by 6), and c by $\frac{1}{2}$ the Arc A (by 39)

42. An $\angle$ in a Segment $>, =, <$ Semicircle is Acute, Right, Obtuse: For 'tis subtended by an Arc $<, =, >$ Semicircumference, whose $\frac{1}{2}$ (the Measure of the $\angle$, by 39) is $<, =, >$ 90d. and therefore is Acute, Right, Obtuse (by 7)

43. Whence, To draw a Tangent (PT) to any point (P) of the Circumference of a Circle: Let PB = Rad. (PC), draw CB out, let BT = BC, draw TP the Tangent sought.

For since PB = BC = BT (by Constr.) an Arc passing thro' C, P, T, (by 19) will be a $\frac{1}{2}$ Circumference, th. $\angle P = \angle$, (by 42) th. PT is a Tangent to the point P (by 12)



Hence the Practical Methods of Erecting and Letting fall Perpendiculars are evident.

44. Opposite $\angle$ s ($n + N$) in the Circumference, standing on the same Chord are $= 2 \angle$ s. For each is measured by $\frac{1}{2}$ its subtending Arc, therefore both by $\frac{1}{2}$ the Circumference, which measures $2 \angle$ s (by 7)



45. Hence, The op. $\angle$ s of a Quadrilateral Fig. inscr. in a Circle are $= 2 \angle$ s. And that Quaadrilat. Fig. the Sum of whose op. $\angle$ s is $= 2 \angle$ s, may be inscrib'd in a Circle.

46. And the Sides thereof be produc'd, the external $\angle$ (a) is $=$ to the internal op. one (N.) For a ($+n = 2 \angle$ (by 9) $= N$ ($+n$ by this.)

47. The $\angle$ (x, z,) made by a Tangent and Chord, is $=$ so an $\angle$ (u, o) in the op. Segment.

For x, z, $= \frac{1}{2}a$, $\frac{1}{2}A$ (by 38) $= u, o$ (by 39)



48. Hence

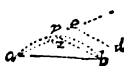
48. Hence, To cut off a Segment from a given Circle capable of containing a given L (z).

Draw the Tangent ab (by 43,) make $Lx = Lz$, then $a = x$ (by 47) $= z$, therefore rac is the Segment reqd.



49. Also, having the Chord (ab) of a Segment capable of containing a given L (x): To find the Point (p) thro' which the Arc shall pass.

Draw ae , and at any point (e) make $Le = Lx$, draw $bp \parallel de$, then $p (= e) = x$, therefore the Arc shall pass thro' p : In the same manner the other points are found.



50. In a Circle, equal Chords are equally distant from the Centre; and the Contrary. For those $=$ Chords are $=$ double Sines, therefore their Co-sines, which is their distance from the Centre, must be $=$.

51. In a Triangle, the $\frac{1}{2}$ of each Side is the Sine of its op. L . For (being inscr. in a Circle by 19) the Sides are Chords of Arcs, which measure $\frac{1}{2}$ their op. L s; And $\frac{1}{2}$ those Chords are the Sines of those Arcs (by 8) i. e. $\frac{1}{2}$ the Sides are the Sines of their op. L s.

52. The 3 L s of every Plane Δ , is $= 2 L$ s: For (being inscr. in a Circle by 19) their Measure is $\frac{1}{2}$ the Circumference, which measure $2 L$ s (by 7)

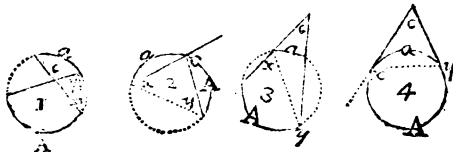
53. Therefore in a Plane Δ , if the Side be produced, the extern. L (d) will be $=$ to the two intern. and op. ones ($a + b$.) For $a + b (+c = 2 L$ s by 52) $= d (+c$ by 9)



54. Hence the measure of an Angle, as c in Figure

1. Neither at Centre or Circumf. made by two Crossing Chords; Or.

2. In the Circumf. made by the Chord and Secant; is $(= \frac{1}{2} A + x$ by 53) $= \frac{1}{2} A + \frac{1}{2} a$ (by 39)

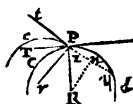


3. With-

3. Without the Circumference, made by two Secants; or

4. Made by two Tangents; is $(= x - y$ by 53) $= \frac{1}{2}A - \frac{1}{2}a$ (by 39)

55. The mixt $\angle$ RPd made by the Circumference, and Rad. is greater than any Rectilinear Acute $\angle$ (RPy or z.) For drawing Rn Perp. Py, then $\angle n (= \angle) > \angle z$. therefore RP $>$ Rn, and the point n is within the Circle Conſeq. $\angle z < \angle$ RPd.



Whence, the $\angle$ made by the Circumference and Tangent is less than any Acute $\angle$.

Therefore no Right Line can be drawn between the Tangent and Circumference, tho' an infinite Number of Curv'd ones may.

56. A Curvilinear $\angle$ (CPc) made by two Intersecting Circumferences (C, c) is $=$ to a Rectilinear $\angle$ (RPr) made by the Radii (RP, rP) drawn to the Intersect. point (P.) For, drawing the Tangents PT, Pr, then $\angle$ PCP $= \angle$ PT, and $\angle$ PrT $(+ \angle$ TPr $= \angle) = \angle$ RPr $(+ \angle$ rPT.)

57. In a Triangle $>, =, <$ Angles are subtended by $>, =, <$ Sides; And the contrary. For $>, =, <$ Sides are Chords of Arcs which measure $>, =, <$ Angles.

1. Therefore Equilateral Δ s must be also Equiangular; And the contrary.

2. And those Δ s are equal, if in both, either 3 Sides; or 2 Sides, with the included $\angle$; or 2 Sides, with the $\angle$ op. to the same Side; or 2 $\angle$ s, with the interjacent Side; or 2 $\angle$ s, with the Side subtending the same $\angle$; be equal.

3. Therefore the $\angle$ s at the Base of an Isosceles Δ are $=$; For the equal Sides are Chords of Arcs measuring $= \angle$ s.

4. And that Line from the Vertex bisecting the Base of an Isosceles Δ , is Perp. to it.

58. The Intern. $\angle$ s of any Polygon are $=$ to twice as many $\angle$ s as it has Sides, except 4. For (n being the N o of Sides or $\angle$ s,) every Polygon may be divided into $n-2$ Δ s, (by Lines drawn from any $\angle$ to all the rest, except the adjoining ones,) therefore will have $2n-4 \angle$ s.

H h

1. And

1. And (producing the Sides) all the extern $Ls = 4 Ls$.
 For each Intern with its Extern is $= 2 Ls$, (by 9)
 therefore all the Intern and Extern $Ls = 2 n Ls$; but
 the Intern $Ls = 2 n L - 4 Ls$ (by 58) th. the Extern
 $Ls = 4 Ls$.

2. Therefore the Intern L (A) of any Regular Polygon

$$is = 2 n - 4 \div n L; \text{ For } n A = 2 n - 4 Ls$$

3. And the L (a) at the Centre is $= 4 Ls \div n (= 2 Ls - A)$

59. Hence it follows, that there are only 3 Regular Surfaces that can fill a Space, viz. Triangles, Squares, and Hexagons.

Since all the Ls round a Point $= 4 Ls$, and the L of

a $\left\{ \begin{array}{l} \text{Regul. } \Delta \text{ is } \frac{2}{3} L \\ \text{Square is } 1 L \\ \text{Hexagon is } \frac{4}{3} L \end{array} \right\}$ which multiply by $\left\{ \begin{array}{l} 6 \\ 4 \\ 3 \end{array} \right\} = 4 L$

But in a Pentagon, 3 Ls are less, 4 Ls greater 4 Ls , and in Figures having more Sides, 3 Ls greater 4 Ls , therefore, &c.

60. Also, There are only 5 Regular Bodies. For a Solid L consists of less than 4 Ls , and 3 Plane Ls are the fewest that can make it; But 6, 4, 3 Ls of Δs , $\square s$, Hexagons, make 4 Ls , and 4 Ls of Pentagons are greater 4 Ls , therefore only a Δ , $\square$, and a Pentagon. can form a Solid L ; so that there can be but 5 Regular Bodies, viz. the

Tetr.	} adrum of	{	4	} Solid Ls	made of	{	3 Δs	} And has	{	4	} Plane Sides	{	6	} Lin. Edges	{	12	} Plane Ls
Oct.			6				4 Δs			20			12			24	
Icos.			12				5 Δs			6			30			60	
Hex.			8				3 $\square s$			12			12			24	
Dodec.			20				3 $\square s$			12			30			60	

61. The Intersection of 2 Planes is a Right Line: For the Right Line drawn on either Plane from any two Points of the Section will be the same. And

1. The Intersections of 2 Paral. Planes by a 3d are Paral.
2. A Perp. to crossing Lines, is Perp. to their Planes.

3. Right

3. Right Lines, or Planes, Perp. to the same Plane, or Right Lines are Parallel.

4. A Perp. to a Plane is Perp. to its Paral. Plane; and the contr.

5. If a Right Line (L) be Perp. to any Plane (P) then all Planes passing thro' (L) are Perp. to P; And if 2 Planes be Perp. to P, their Interfection will also be Perp. to P.

6. In any Number of Planes passing thro' the Line L. Paral. to a certain Plane P, their Interfection with P are $\parallel$ to L, and to each other.

7. Two pair of meeting Lines respectively $\parallel$ in different Planes contain $\equiv$ Ls, and Paral. Planes.

62. In Equiangular Δ s, the Sides about $\equiv$ Ls are directly $::1$; And the contr. For, being inscr. in a Circle, the Sides will be Chords of Similar Arcs.

63. Therefore Equiangular Δ s are similar (by 21); Conseq. similar Δ s have their Sides about $\equiv$ Ls $::1$.

1. And therefore those Δ s are also similar, which have an L $\equiv$, and the Sides about that L $::1$; Or which have 2 Sides $::1$, an L op. to one of 'em $\equiv$, and the other of the same kind.

2. Hence similar Δ s have their Heights $::1$ to their Bases; For making a Side next the vertical L Radius, the Heights will be Sines of similar Arcs, which are as the Radii (by 31.) and those Radii as the Bases (by 62).

3. And a Right Line, or Plane cutting a Triangle, or Pyramid, either parallel or subcontrary to the Base, cuts off a Figure similar to the whole. For the L at the Top is common, the others are $\equiv$ by Constr. therefore the Figures are similar.

4. Conseq. in a Δ , the parts of the Sides cut by a Right Line drawn $\parallel$ to its Base are directly $::1$; And the contr. For the Δ cut off being Sim. to the whole (by Constr.) the Sides about the common L at the Top will be $::1$ (by 62.) therefore, &c.

5. Hence is drawn the Method of dividing a Right Line in 2 given $::$; And of finding a 3d, or 4th $::1$ to 2, or 3 given right Lines; with Variety of Practices in Geometry.

6. In any Δ , the Base is to its Parallel, as the Sides are to the Parts next the Vertex.

For $y : x :: n : m :: b : p$ (a) by 4. 63

Th. $y + x : x :: n + m : m :: b + a : p$



7. In any Quadrilat. Fig. inscrib'd in a Circle, the $\square$ of the Diagonals is = to the 2 $\square$ s of the op. Sides.

Let $\angle DAE = \angle BAC$, $AC = y$, $BE = m$; then Δ s AED, ABC, and ABE, ACD, are Similar.

Th. $\left\{ \begin{matrix} b : n :: y : c \\ a : m :: y : d \end{matrix} \right\}$ by 63, th. $\left\{ \begin{matrix} bc = ny \\ ad = my \end{matrix} \right\}$

Th. $ad + bc = (my + ny) y \times m + n$



8. All Figures of the same kind, standing upon the same, or equal Bases, and of = Altitudes are =. For the Elements of the one are respectively equal to those of the other, both in Number and Magnitude. Th. in Pgrs. Ppds. Prisms, and Cylinders, if B, H, be the Base, and Height, then $BH = \text{Content}$: And in Triangles, the Area $= \frac{1}{2} BH$; bec. Triangle : Circumscr. Pgr. :: 1 : 2 (by this).

9. And a Rectilinear Figure of n Sides may be reduced into $n-2$ Triangles, by $n-3$ Diagonals; and its Area is = to the Area of all its Triangles.

10. Th. in a Circle whose Area call A, Circumf. c , Rad. r , since $\frac{1}{2} r c$ ($\frac{1}{2} r c^\circ c$) = A, th. $\frac{1}{2} r c = A$, (by 36.) And $d^2 (4r) : \frac{1}{2} r c :: 2r (d) : \frac{1}{2} c$. Also, if $c = \text{Arc}$, then the Sector $= \frac{1}{2} r c$.

11. All Homologous Figures, (1) Of equal Bases are as their Heights, or of equal Heights, are as their Bases; (2) Of different Bases and Heights, are as their Products, or those of the Sides into the Bases or Heights; (3) And theref. are also as their like Sides raised into the Figure's Dimension.

Suppose L, l, the Length or Sides of the Figures (x, y) or let S, rS, φ S, the Sides of the one, and $s, rs, \varphi s$, those of the other; then, if

If $\left\{ \begin{matrix} B = b, \text{ or } H = b; \\ \text{all differ;} \end{matrix} \right. x : y :: BH : bb :: H : b, \text{ or } :: B : b$.

For $B : b :: H : b$ (by 2.63) :: $L : l$ (by 63) Therefore

In

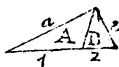
In $\begin{cases} \text{Planes } x:y :: (S^2r:S^2r ::) S^2:S^2. \\ \text{Solids } x:y :: (S^3r:S^3r ::) S^3:S^3. \end{cases}$

And like inscrib'd Figures are as their respective wholes, Th. all similar Surfaces, or Solids, are as the Squares, or Cubes of their like Sides.

And if such Figures are equal, they have their Bases and Heights reciprocally :: 1. And the contrary.

12. A Right Line bisecting an $\angle$ of a Δ , cuts the op. Side directly :: 1 to the Legs. i.e.

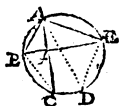
$y:z :: A:B$ (by 11.63) :: $ax:ex$ (by 26) :: $a:e$.



13. If from any $\angle$ (BAE) of a Δ Inscr. in a Circle, whose Diameter is AD, a Perp. (AF) be let fall upon the op. Side, then (bec. Δ ABF Sim. AED) $AF:AB :: AE:AD$.

Also, let AC bisect $\angle A$, then (bec. Δ ABC Sim. AEF) $AC:AB :: AE:$

AF , or $BAE = CAF = AF^2 + CFA = AF^2 + BFE$.



In any Plane Δ , whose Sides are a, b, c , Height p , and their op. Ls x, y, z . Since the Diameter (of the Circumscr. Circle) $d :: a :: c :: (ac \div d) p$. th. Area $A = acb \div 2d$, th. $d :: \frac{1}{2}a :: cb:A$, or $d:s, Lx :: cb:A$.

Again, Δs having one $\angle$ common, are as the Falls of the Sides about that $\angle$.

Also, since $d:c :: (ab:pb = 2A ::) \frac{1}{4}ab :: \frac{1}{2}A$, or $\frac{1}{2}d:c :: \frac{1}{4}ab:A$, th. $\frac{1}{2}cd:c^2 :: \frac{1}{4}ab:A$, that is, $dxs, z:c^2 :: s, x \times s, y:A$.

64. A Perpendicular demitted from the Right $\angle$ of a Plane Δ upon the op. Side, divides that Δ into two, each similar to the whole, and therefore to one another. For each has 1 $\angle$, and 1 $\angle$ common to the whole, therefore are similar (by 63.) Whence



1. $\begin{cases} a:x :: x:b \\ c:y :: y:b \end{cases}$ by 63, $\begin{cases} \text{th. } x^2 = ab \\ \text{th. } y^2 = cb \end{cases}$

2. Th. $x^2 + y^2 (= ab + cb = a + c \times b) = b^2$

3. In a $\angle \Delta$, $b = \sqrt{x^2 + y^2}$, $x = \sqrt{b^2 - y^2}$, $y = \sqrt{b^2 - x^2}$

4. And

4. And $x^2 : y^2 (:: ab : cb) :: a : c$.

5. $x^2 : p^2 (:: ab : ac) :: b : c$, & $y^2 : p^2 (:: cb : ac) :: b : a$

6. $b + y : x :: x : b - y$; And $b + x : y :: y : b - x$.

7. $a : p :: p : c :: x : y$, th. $p^2 = ac$; $ay = px$; $ax = py$.

8. Th., To find a mean z between any 2 Right Lines a, c .

Let $a + c$ be made the Diameter of a Circle, the Perpendicular on the Point of meeting, and terminated by the Circumf. will be the mean reqd.

9. Whence we have the Method of making a Square = to a given Pgr. And th. = to any Right lin'd Figure given.

10. Also, To cut a Line (l) in Extrem and Mean ::; (i.e. that the $\square$ of the greater part (x) may be = to the $\square$ made of the lesser part (l - x) and the whole Line.)

Make l and $\frac{1}{2}l$ the Legs of a $\triangle$; then the Hypotenuse $-\frac{1}{2}l = x$ sought. For $x^2 = l^2 - lx$, th. $x^2 + lx = l^2$, th. $x = l^2 \div \frac{1}{2}l^2 = \frac{1}{2}l$.

11. And, In any Plane $\triangle$, the Sides being b, x, y : Sup. lines (m, m) from the Vertex v , making each at the Base b a like $\angle$ with that at v ; let x be the part of the Base line intercepted between m and m , Then $b^2 = x^2 + y^2 \pm \frac{2}{m}xy$, if $v >$, or < 90 degrees.

Theref. if $v = 90^\circ$, $b^2 = x^2 + y^2$; If $v = 120^\circ$ or 60° , $b^2 = x^2 + y^2 \pm xy$:

Hence the Radius of any Circle is = Chord of 60° .

If $v = 135^\circ$, or 45° , $b^2 = x^2 + y^2 \pm xy \times \sqrt{2}$

If $v = 150^\circ$, or 30° , $b^2 = x^2 + y^2 \pm xy \times \sqrt{3}$, &c.

12. In a Circle, the Circumscrib'd Square is to the Inscrib'd as 2 to 1. For the (Diameter, or) Side of the Circumscr. is the Diagonal of the Inscrib'd Square; but the Diagonal squared is to the Side squared, as 2 to 1.

Hence

13. Hence the Diagonal (D) is Incommensurable to the Side (S) of the Square. For $S:D::1:\sqrt{2}$, and (s) any part of S shall be in the same :: to d, its corresponding part of D; and how small soever s be taken, yet d is capable of a further Division, and that in infinitum; therefore 'tis impossible to find such a part s, that will exactly measure D.

Whence the Infinite Divisibility of Quantity is evident.

14. If similar Figures (H, X, Y) be made on the Sides (h, x, y) of a $\triangle$; that (H) on the Hypotenuse (h) is $\equiv$ those (X, Y) on the Legs (x, y.) For H, X, Y are as b^2, x^2, y^2 , (by 11.63) But $b^2 = x^2 + y^2$, (by 2.64) th. $H = X + Y$.

Hence 'tis easie to find the Difference between any similar Figures; As also, the Sum of any Number of such Figures.

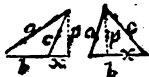
And the Quadrature of the Lunes of Hippocrates of Scio, is also evident. Let abc be an isosceles $\triangle$, $\angle$ at a. Then $m + r = x + r$ (by 4.64) th. $m = x$.



15. Also similar Figures made on 4::1 Lines (a, b, c, d) are :: 1. As sup. Sim. $\triangle$ s A, B on a, b; and Sim. Pentagon. C, D on c, d; then $A:B::a^2:b^2$ (by 11.63) $::c^2:d^2$ (by Proport.) $::C:D$ (by 11.63)

16. In Obtuse, or Acute $\angle$ d $\triangle$ a. $c^2 = a^2 + b^2 \pm 2bx$.

For ($p^2 =$) $a^2 - b^2 \pm 2bx + x^2$
 $= c^2 - x^2$



17. Hence $x = \frac{a^2 + c^2 + b^2}{2b}$

18. Also, given the 3 Sides of a Plane $\triangle$, reqd. the Area. (Fig. 2.) Since $p^2 = c^2 - x^2 = a^2 - b^2 \pm 2bx - x^2$, th.

$x = \frac{c^2 + b^2 - a^2}{2b}$, & $p = \frac{c^2 - x^2}{\frac{1}{2}}$ th.

$(\frac{1}{2}b \times p \text{ or }) \frac{1}{4}b^2 \left| \frac{1}{2} \times c^2 - c^2 + b^2 - a^2 \div 2b \right| \frac{1}{2}$ or

$\frac{1}{8}a^2b^2 + a^2c^2 + b^2c^2 - \frac{1}{16}a^4 - b^4 - c^4 \left| \frac{1}{2} \right.$ or

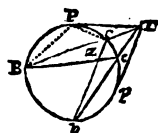
(If $b = \frac{1}{2}a + b + c$) $b \times b - a \times b - b \times b - c \left| \frac{1}{2} \right. = \text{Area.}$

19. A

19. A Tangent (PT) is a mean :: 1 between the whole Secant (BT) and its extern. Part (CT.)

For $\triangle TPC$ Sim. $\triangle BTP$, therf. BT:

$$PT :: PT : CT, \text{ or } \overline{PT}^2 = BTC.$$



20. Whence the $\square$ s made of such Secants, from the same Point (T) and their outward Segments are =. . e. $BTC = bTc$, For each is = to the same $\square$ (PT^2 .)

21. Therf. those Secants are reciprocally as their outward Segments; i. e. $BT : bT :: cT : CT$.

22. Hence also, Two Tangents drawn from the same point are =: i. e. $PT = pT$; For the Sq. of each is = to the same Rectangle.

23. And the parts of Crossing Chords are reciprocally :: 1: i. e. $Bz : Cz :: bz : cz$, or $Bzc = bzC$; For $\triangle BzC$ Sim. $\triangle bzC$.

65. Hence if $r = \text{Rad.}$ $d = \text{Diameter}$, $s = \text{Sine}$, $v = \text{Versed-sine}$, $v = \text{Co-versed-sine}$, $c = \text{Chord}$, $s = \text{Co-sine}$, $t = \text{Tangent}$, $t = \text{Co-tangent}$, $f = \text{Secant}$, $f = \text{Co-secant}$, $a = \text{Arc}$.

$$1. dv - vv = s^2 = 2rv - v^2; \text{ for } d - v : s :: s : v.$$

$$2. d : c :: c : v; \text{ for } dv (\equiv v^2 + s^2) = c^2.$$

$$3. C^2 : c^2 (:: dV : dv) :: V : v.$$

$$4. d : C + c :: C - c : V - v, \\ \text{for } C^2 - c^2 = dV - dv$$

$$5. st = r^2; \text{ for } BT \times DE = BCD$$

$$6. sf = r^2; \text{ for } Af \times CE = ACD$$

$$7. sf = r^2; \text{ for } Cf \times CT = BCA$$

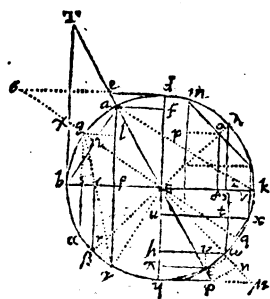
$$8. \text{Th. } t, A \times t, A (\equiv r^2) = t, a \times t, a$$

$$9. \text{And } s, A \times f, A (\equiv r^2) = s, a \times f, a$$

$$10. sr = st; \text{ for } Af \times CB = Cf \times BT$$

$$11. \begin{cases} rv = s, \frac{1}{2}a \times 2s, \frac{1}{2}a; \text{ for } CB_f = BN \times BA \\ rv = s, a \times t, \frac{1}{2}a \text{ for } CB_f = Af \times Bz \end{cases}$$

$$12. \text{Th. } \frac{1}{2}rv = s^2, \frac{1}{2}a = \frac{1}{2}s, a \times \frac{1}{2}a$$



13. And

13. And $L, v = 2 L, s, \frac{1}{2} a - L, s, 30^\circ$

14. $\begin{cases} \frac{1}{2} r v = s^2, \frac{1}{2} a; \text{ for } \frac{1}{2} K_F \times CB = CN. (\frac{1}{2} AK) \times CN \\ \frac{1}{2} r v = \frac{1}{2} s, a \times t, \frac{1}{2} a; \text{ for } \frac{1}{2} K_F \times CD = \frac{1}{2} A_F \times D_F \end{cases}$

15. $\begin{cases} r \times s, 2a = 2s, a \times s, a; \text{ for } CB \times A_F = AB \times CN \\ r \times v, 2a = 2s^2, a; \text{ that is, } CB \times B_F = AB \times BN \end{cases}$

16. Also, $s, a +$ or $-t, a = t$ or $t, \frac{1}{2} 90^\circ - a$

17. And 'tis evident, $C_e \times (V_e) H\pi = \pi e \times e\omega$, for $\Delta e V\omega$ Sim. $\Delta e \pi C$; But taking the Tangent $e\omega$ infinitely small, 'twill be = its corresponding Arc, th. $s \times e\omega = r \times H\pi$; Conseq. the Sum of all the Sines erected on any Arc (eY) is $= rv$: Th. the Sum of all the Sines erected on the Quadrantal Arc is $= r^2$.

18. Also, if $e\omega$ be infinitely small, then $C\phi : CY :: \phi\mu$ (t) ϕn , and $C\phi : C_e :: \phi n : e\omega(a)$, Th. $r^2 : r^2 :: t : a$

19. $Ca : CO :: as : P_i (Od)$; & $Ca : Cs :: MO : MP$
Th. $(P_i \pm MP) s, A \pm a = s, a \times s, A \mp s, a \times s, a \div r$

20. If, $kq = 30^\circ$, and $qx = q\omega$, then $yq = 60^\circ$. And $\omega z = xv + \sqrt{4 qx^2}$ (or $4 \times \frac{1}{2} \omega x^2 = 4 xt^2$) $- xt^2 (qx^2)$ or $\sqrt{3} qx^2$, or $\sqrt{3} \times qx$, i. e. s , Arc $> 30^\circ = s$, Arc as much $< 30^\circ + s$, of its Defect $\times \sqrt{3}$.

21. Also $ux = h\omega + \omega q$, i. e. s , Arc $> 60^\circ = s$, Arc as much $< 60^\circ + s$, of its Defect.

22. If $Ba, B\beta, B\gamma$, be Equidiff. Arcs, which call a, m, e ; their common Diff. d , then $C\gamma : CR :: \gamma G (2\tau\beta) :: \gamma L : \gamma\gamma + sa$, i. e. $r : s, d :: 2s, m : s, e + s, a = r$

Th. $s, m : s, a + s, e :: s, M : s, A + s, E$

And $s, a + s, e = s, d \times 2s, m$ or $2 s, d \times s, m \div r$

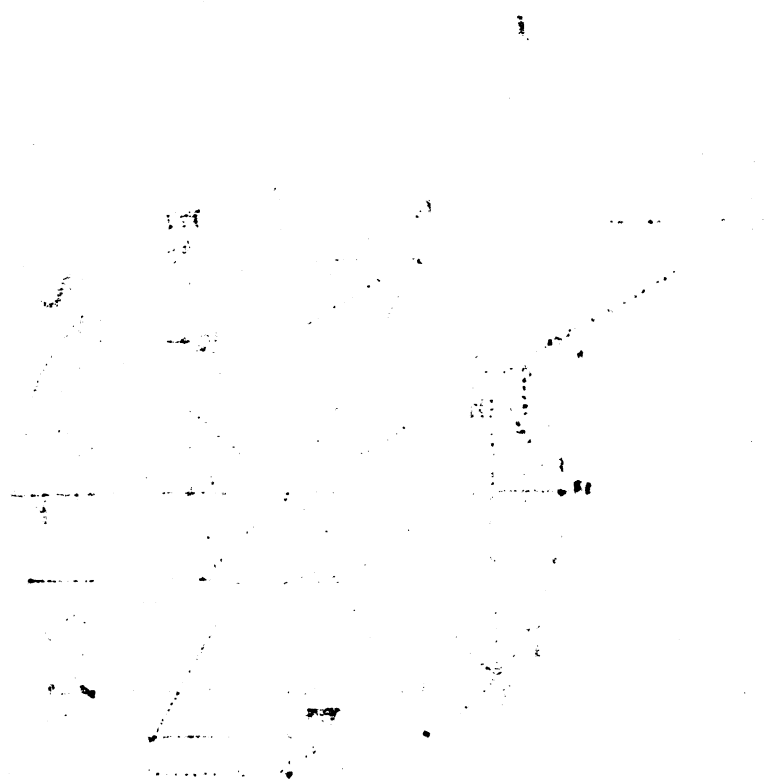
23. Because $n = S - s$, $mo = \frac{1}{2} \text{Chord of } A - a$, then op , produc'd to dc , will be $= \frac{1}{2} S + s$; Th. $d (V - v) : S - s :: \frac{1}{2} S + s : S^2 - s^2 \div \frac{1}{2} d (q)$, and $q \pm \frac{1}{2} d = s, a$, or s, A

24. Since $r^2 - s^2 |^{\frac{1}{2}} (s) : r :: s : (a) r^2 - s^2 |^{-\frac{1}{2}} \times r : s$

Th. $a = \frac{s}{1} + \frac{1 \times s^2}{2 \times 3 r^2} a + \frac{9 \times s^2}{4 \times 5 r^2} \beta + \frac{25 \times s^2}{6 \times 7 r^2} \gamma + \&c.$

I i

25. And



13. And $L, v = 2 L, s, \frac{1}{2} a - L, s, 30^\circ$

14. $\left\{ \begin{array}{l} \frac{1}{2} r v = s^2, \frac{1}{2} a; \text{ for } \frac{1}{2} K_e \times CB = CN. (\frac{1}{2} AK) \times CN \\ \frac{1}{2} r v = \frac{1}{2} s, a \times t, \frac{1}{2} a; \text{ for } \frac{1}{2} K_e \times CD = \frac{1}{2} A_e \times D_e \end{array} \right.$

15. $\left\{ \begin{array}{l} r \times s, 2a = 2s, a \times s, a; \text{ for } CB \times A_e = AB \times CN \\ r \times v, 2a = 2s^2, a; \text{ that is, } CB \times B_e = AB \times BN \end{array} \right.$

16. Also, $s, a \div$ or $-t, a \div$ or $t, \frac{1}{2} 90^\circ - a$

17. And 'tis evident, $C_e \times (V_e) H\pi = \pi s \times \varepsilon\omega$, for $\Delta e V\omega$ Sim. $\Delta e \pi C$; But taking the Tangent $\varepsilon\omega$ infinitely small, 'twill be = its corresponding Arc, th. $s \times \varepsilon\omega = r \times H\pi$; Conſeq. the Sum of all the Sines erected on any Arc (εY) is $= r v$: Th. the Sum of all the Sines erected on the Quadrantal Arc is $= r^2$.

18. Also, if $\varepsilon\omega$ be infinitely ſmall, then $C\phi : CY :: \phi\mu$

(t) ϕn , and $C\phi : C_e :: \phi n : \varepsilon\omega (a)$, Th. $s^2 : r^2 :: t : a$

19. $C\lambda : CO :: \lambda s : P_t (O\delta)$; & $C\lambda : C_s :: MO : MP$
Th. $(P_t \pm MP) s, A \pm a = s, a \times s, A \pm s, A \times s, a \div r$

20. If, $kq = 30^\circ$, and $qx = q\omega$, then $yq = 60^\circ$. And

$\omega z = xv + \sqrt{4qx^2}$ (or $4 \times \frac{1}{2} \omega x^2 = 4xt^2$) $- xt^2 (qx^2)$
or $\sqrt{3} qx^2$, or $\sqrt{3} \times qx$, i. e. s , Arc $> 30^\circ = s$, Arc as much $< 30^\circ + s$, of its Defect $\times \sqrt{3}$.

21. Also $ux = hw + \omega q$, i. e. s , Arc $> 60^\circ = s$, Arc as much $< 60^\circ + s$, of its Defect.

22. If $B\alpha, B\beta, B\gamma$, be Equidiff. Arcs, which call a, m, e ; their common Diff. d , then $C\gamma : CR :: \gamma G (2\tau\beta) :: \gamma L : \gamma\gamma + sa$, i. e. $r : s, d :: 2s, m : s, e + s, a = r$

Th. $s, m : s, a + s, e :: s, M : s, A + s, E$

And $s, a + s, e = s, d \times 2s, m$ or $2 s, d \times s, m \div r$

23. Becauſe $n = S - s$, $mo = \frac{1}{2}$ Chord of $A - a$, then op , produc'd to dc , will be $= \frac{1}{2} S + s$; Th. $\delta (V - v)$
 $S - s :: \frac{1}{2} S + s : S^2 - s^2 \div \frac{1}{2} \delta (q)$, and $q \pm \frac{1}{2} \delta = s, a$, or s, A

24. Since $r^2 - s^2 |^{\frac{1}{2}} (s) : r :: s : (a) r^2 - s^2 |^{-\frac{1}{2}} \times r s$

Th. $a = \frac{s}{1} + \frac{1 \times s^2}{2 \times 3 r^2} \alpha + \frac{9 \times s^2}{4 \times 5 r^2} \beta + \frac{25 \times s^2}{6 \times 7 r^2} \gamma + \&c.$

I i

25. And

$$25. \text{ And } s = \frac{a}{1} - \frac{a^3}{1 \times 2 \times 3 r^2} + \frac{a^5}{1 \times 2 \times 3 \times 4 \times 5 r^4} - \dots, \&c.$$

$$26. \text{ Rec. } dv = v^2 |^{\frac{1}{2}} (s) : r :: v : (a) \quad dv = v^2 |^{\frac{1}{2}} \times r \bar{v}$$

$$\text{Th. } a = 1 + \frac{v}{6} + \frac{3v^2}{40d} + \frac{5v^3}{112d^2} + \frac{35v^4}{1152d^3}, \&c. \times dv |^{\frac{1}{2}}$$

$$27. v = \frac{a^2}{1 \times 2 r} - \frac{a^4}{1 \times 2 \times 3 \times 4 r^3} + \frac{a^6}{1 \times 2 \times 3 \times 4 \times 5 \times 6 r^5}, \&c.$$

$$28. \text{ Th. } s(r-v) = r - \frac{a^2}{1 \times 2 r} + \frac{a^4}{1 \times 2 \times 3 \times 4 r^3} - \dots, \&c.$$

$$29. f(1 \div s) = 1 + \frac{1}{2}a^2 + \frac{1}{24}a^4 + \frac{61}{720}a^6 + \frac{277}{8064}a^8 + \dots, \&c.$$

$$30. \text{ Th. } S, f's = a + \frac{1}{2}a^3 + \frac{1}{24}a^5 + \frac{61}{5040}a^7 + \frac{277}{72576}a^9, \&c. \\ = L, t, \frac{1}{190} + a \text{ in Naper's Form.}$$

$$31. f(1 \div s) = 1 \div a + \frac{1}{2}a + \frac{7}{720}a^3 + \frac{31}{11520}a^5 + \dots, \&c.$$

$$32. \text{ Rec. } \frac{r^2}{s} = f, \& \frac{rs}{s} = a, \text{ also } (af) \frac{r^3 s}{r^2 - s^2} = F, S, f's$$

$$= s + s s^2 + s s^4 + s s^6 + s s^8 + s s^{10} + \dots, \&c. (\text{if } r=1)$$

$$\text{Th. } S, f's = s + \frac{1}{3}s^3 + \frac{1}{5}s^5 + \frac{1}{7}s^7 + \frac{1}{9}s^9 + \frac{1}{11}s^{11}, \&c.$$

$$\text{Also } S, t's = \frac{1}{2}s^2 + \frac{1}{4}s^4 + \frac{1}{6}s^6 + \frac{1}{8}s^8 + \frac{1}{10}s^{10}, \&c.$$

$$33. \text{ And (if } r=1) \bar{a} = t \div 1+t^2 \text{ (by 18, 65)} =$$

$$t \times 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + t^{12} \&c.$$

$$\text{Th. } a = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \frac{1}{11}t^{11} + \dots, \&c.$$

$$34. \text{ And } t = a + \frac{1}{3}a^3 + \frac{1}{5}a^5 + \frac{1}{7}a^7 + \frac{1}{9}a^9 + \dots, \&c.$$

$$\text{Th. } t(1 \div t) = 1 \div a - \frac{1}{3}a + \frac{1}{5}a^3 - \frac{1}{7}a^5 + \frac{1}{9}a^7 - \dots, \&c.$$

$$35. S, t's = \frac{1}{2}a^2 + \frac{1}{12}a^4 + \frac{1}{45}a^6 + \frac{1}{2520}a^8 + \frac{1}{110790}a^{10}, \&c.$$

$$36. \text{ And if } n = 2, 3058509, \&c. (10 \text{ being Radius}) \text{ then} \\ n \pm \frac{1}{2}a^2 \pm \frac{1}{12}a^4 \pm \frac{1}{45}a^6 \pm \frac{1}{2520}a^8, \&c. = L, f, \text{ or } L, s, \text{ of } a, \\ \text{in Naper's Form; And } \frac{1}{n} \text{th thereof will give Briggs's.}$$

$$37. \text{ Given } c \text{ the Chord of an Arc } a: \text{ Reqd. } G \text{ that of} \\ \text{another Arc } A, \text{ so that } A:a::n:1. \text{ Since}$$

$$a = t$$

$$c = c + \frac{c^3}{6d^2} + \frac{3c^5}{40d^4}, \&c. A = C + \frac{C^3}{6d^2} + \frac{3C^5}{40d^4}, \&c. \text{ (by 24)}$$

$$\text{Th. } C + \frac{C^3}{6d^2} +, \&c. = n \times c + \frac{c^3}{6d^2} +, \&c. = A$$

$$\text{Th. } C = \frac{nc}{1} + \frac{1-n^2}{2 \times 3d^2} c^3 \alpha + \frac{9-n^2}{4 \times 5d^2} c^5 \beta + \frac{25-n^2}{6 \times 7d^2} c^7 \gamma, \&c.$$

$$38, \text{Bec. } t, a = \left(\frac{rs}{s} \right) \frac{rs}{\sqrt{r^2 - s^2}} = (\text{if } a \text{ be } 30^\circ) \frac{1}{\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}}$$

$$\text{And } 6a, \text{ or } 6 \times t = \frac{1}{3} t^2 + \frac{1}{3} t^5, \&c. = \frac{1}{3} \text{ Periphery } (\pi)$$

$$\text{But } 6 \times \frac{1}{\sqrt{3}} = \frac{\sqrt{36}}{\sqrt{3}} = \sqrt{12} = 2\sqrt{3}, \text{ and } t^2 = \frac{1}{3}; \text{ Let}$$

$$a = 2\sqrt{3}, \beta = \frac{1}{3}a, \gamma = \frac{1}{3}\beta, \delta = \frac{1}{3}\gamma, \&c.$$

$$\text{Then } a - \frac{1}{3}\beta + \frac{1}{3}\gamma - \frac{1}{3}\delta + \frac{1}{3}\epsilon, \&c. = \frac{1}{3}\pi, \text{ or}$$

$$a - \frac{1}{3}\frac{3a}{9} + \frac{1}{3}\frac{a}{9} - \frac{1}{3}\frac{3a}{9^2} + \frac{1}{3}\frac{a}{9^2} - \frac{1}{3}\frac{3a}{9^3} + \frac{1}{3}\frac{a}{9^3}, \&c.$$

Theref. the (Radius is to $\frac{1}{3}$ Periphery, or) Diameter is to the Periphery, as 1,000, &c. to 3,141592653,58979323 84.6264338327.9502884197.1693993751.0582097494.4592307816.4062862c89.9862803482.5342117067.9 +, True to above a 100 Places; as Computed by the Accurate and Ready Pen of the Truly Ingenious Mr. John Machin: Purely as an Instance of the Vast advantage Arithmetical Calculations receive from the Modern Analysis, in a Subject that has bin of so Engaging a Nature, as to have employ'd the Minds of the most Eminent Mathematicians, in all Ages, to the Consideration of it. For as the exact Proportion between the Diameter and the Circumference can never be express'd in Numbers; so the Improvements of those Enquirers the more plainly appear'd, by how much the more Easie and Ready, they render'd the Way to find a Proportion the nearest possible: But the Method of Series (as improv'd by Mr. Newton, and Mr. Halley) performs this with great Facility, when compared with the Intricate and Prolix Ways of Archimedes, Vieta, Van Ceulen, Metius, Snellius, Lansbergius, &c. Tho' some of them were said to have (in this Case) set Bounds to Human Improvements, and to have left

$$\begin{aligned} \text{I. } s, A + a : s, A - s, a :: s, A + s, a : s, A - a \\ &:: v, A + a : s, A - s, a \\ &:: s, A - s, a : v, A - a \\ &\&c. \end{aligned}$$

$$\begin{aligned} 2. & \quad s_1 A + s_2 a : s_1 A - s_2 a :: t_1 \frac{1}{2} A + a : t_1 \frac{1}{2} A - a \\ 3. & \quad v_1 A - v_2 a : v_1 \frac{1}{2} A + a :: s_1 \frac{1}{2} A - a : \frac{1}{2} r \\ 4. & \quad s_1 \frac{1}{2} A + a : s_2 A + s_3 a :: s_1 \frac{1}{2} A - a : s_1 A - a \\ & \quad \quad \quad (\therefore r : 2s_1 \frac{1}{2} A - \end{aligned}$$

Theref. in *Equidif.* Arcs, (where m is the Mean between a & c , its *Equidist.* Extremes) 'tis,

$$s, m; s, e + s, e :: s, M: s, A + s, E, \text{ (as in Art. 22, 54.)}$$

67. The *Diameter* of a Plane Figure is a Right Line bisecting all Parallels to the Tangents. The Lines bisected are called *Ordinates* to that Diameter. The Point of Concourse of all Diameters is called the *Centre* of the Figure. The Intersection of the Curve and Diameter is called the *Vertex*.

That Diameter, to which the Ordinates stand at right Angles, is called the *Axis*. The common Intersection of the Diameter and Ordinate is called the *Point of Application*. The Segment of the Diameter intercepted between the Vertex and the Point of Application is called the *Abstriss* or *Intercepted Diameter*.

And *Geometric Lines* are distinguished according to the Dimensions of the *Equation* expressing the Relation between the *Ordinates* and *Abscissa*; or according to the Number of Points in which they may be cut by a right Line. Then a *Line* of the *First Order*, will be a *Right Line*; Those of the *Second*, or *Quadratic Order*, will be *Conic Sections*; and those of the *Third*, or *Cubic Order*, will be the *Cubic Paraboloid*, &c. Therefore a *Curve* of the 1st, 2^d, 3^d, &c. Kind, is the same with a *Line* of the 2^d, 3^d, 4th, &c. Order.

And

And a *Line of the Infinitesimal Order*, is that which a right Line may cut in an Infinite Numbers of Points; as the *Spiral, Cycloid, &c.* and all those generated by Infinite Revolutions of a Radius.

68. Of Curves of the First Kind.

If in any Right Line, upon a Plane, there be taken two Points ϕ, f ; an Infinite Number of Points (p) may be found, so that $fp \pm \phi p = vu$ or fl a given Right Line, and $p\phi = pl$ every where. For with vu (or fl) from f , describe the Arc gl , draw the Lines fl , make the $\angle l\phi p = \angle \phi lp$, or $pl = p\phi$; And the Curve drawn thro' p ,

1. If it be $fp \pm \phi p$, is called an *Ellipse, or Hyperbola*; as in Fig. 1. & 2.

2. If f be infinitely distant from ϕ , the Arc gl will become a Right Line perpendicular to vu , and the Curve is called a *Parabola*.

3. If f and ϕ coincide, the Curve will be a *Circle*, whose Radius is $\frac{1}{2}fp + \frac{1}{2}\phi p$.

Therefore, if upon a Plane, any Right Line vu be taken, and therein $v\phi = uf$, and the Points m within, or without ϕ and f ; and Arcs described with vm, um , from ϕ, f , will intersect in the Curve of an *Ellipse, or Hyperbola*.

And in any indefinite Right Line fg (Fig. 3.) let $vg = v\phi$, and pa Perp. fg ; with ga , from ϕ cut pa in p, π , the Curve Line drawn thro' them will be a *Parabola*.

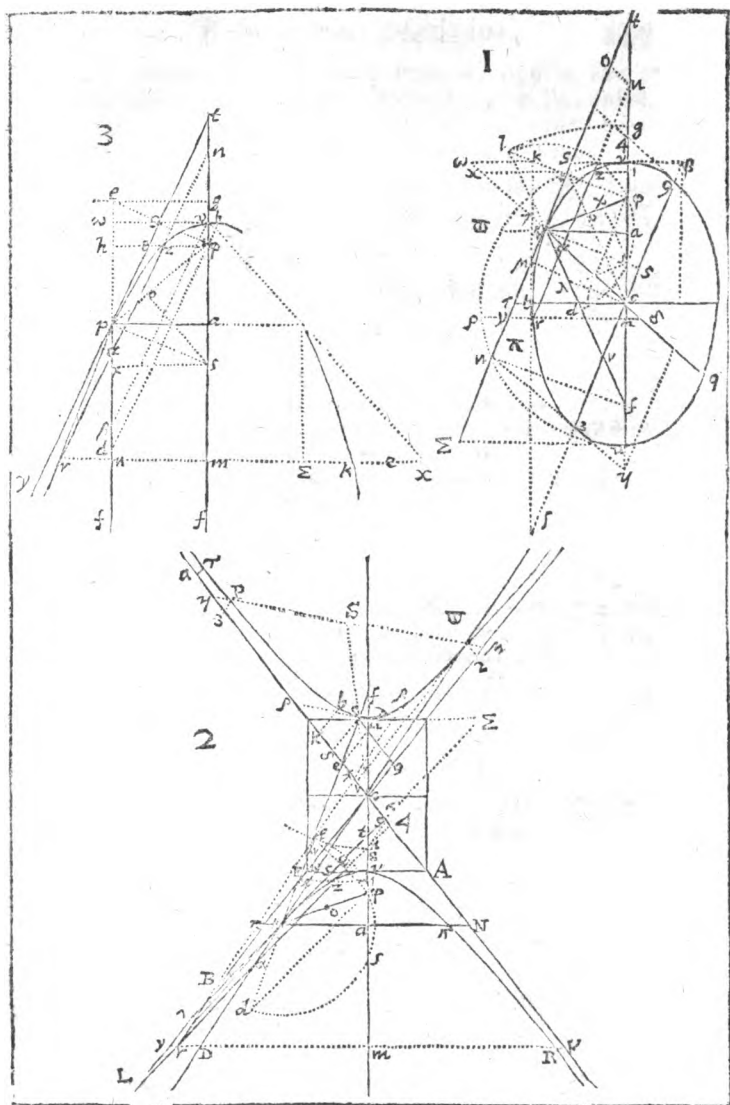
The Point v is the *Vertex*, pa (y) an *Ordinate*, va (x) the *Abscissa*, ϕ, f the *Foci*, vu (t) the *Transverse Axe*.

'Tis evident the *Parabola* has but one *Focus* ϕ , so that t is Infinite. Therefore the Properties of the *Parabolic Curve* are easily drawn from those of the other; yet to gratifie the Learner, we chose to do them separately.

In the Ellipse and Hyperbola.

1. Thro' p draw pt making $\angle lpt = \angle tp\phi$; Then shall pt be a *Tangent* to Curve in p . For, making $fl = p\phi$, and taking any Point e in the Line pt ; then $e\phi = el$, but fl (vu) $< fe + el$ (in the *Ellipse*), and fl (vu) $+ el > fe$ (in the *Hyperbola*). Th. e is without the Curve.

2. Draw



Place this Figure at Page 246.

2. Draw $ce \parallel fp$, cutting ty in e , and qp in o , Then $L \phi ep$ ($Llpe$) = $Lope$. And $eo = po$ = (bec. $qc = \frac{1}{2}of$, and $th. eo = \frac{1}{2}fp$) $\frac{1}{2}pe$, Th. ce ($\frac{1}{2}qp$ & $\frac{1}{2}fp$) = cv . Th. (bec. q, e, p , are in a Semicirc. whose Centre is o) the $L \phi ep = L$.

3. Let ct^2 ($\frac{1}{4}t^2$) : $v\phi u$ ($t \mp q \times q$) :: vu (t) : p = Parameter; Then (bec. $\frac{1}{4}t^2$ ($\frac{1}{4}t \times t$) : $\frac{1}{4}tp$ ($\frac{1}{4}t \times p$) :: $\frac{1}{4}t^2$: $t \mp q \times q$) $\frac{1}{4}tp = t \mp q \times q$, or $v\phi u$.

4. Draw $c9 \parallel tp$, cutting pf in v , Then (bec. $cv \parallel pe$, and $ce \parallel fp$) pv (ce) = cv , (by 2.) Supply the Hyperb. Figure.

5. From p , thro' ϕ describe a Circumf. cutting vu in s , and fp in d , l ; Bisect fd in v ; then

$\frac{1}{2}fl$: $\frac{1}{2}fp$:: $\frac{1}{2}fs$: $\frac{1}{2}fd$, & $e. cv$: cf :: ca : fv , th.

1. cv : $cv + cf$:: ca : $ca + fv$:: $cv + ca$: $cv + cf + ca + fv$.

[that is, cv : fv :: ua : $fp + fa$,] And

2. cv : $cv \cup cf$:: ca : $ca \cup fv$:: $cv \cup ca$: $cv - cf - ca + fv$

[that is, cv : fu :: va : $fp - fa$,]

Th. cv^2 : vfu :: vau : $(fp^2 - fa^2) ap^2$.

i. e. $\frac{1}{4}t^2$: $t \mp q \times q$:: $t \mp x \times x$: y^2 :: t : p :: $t \mp X \times X$: Y^2

6. And (bec. $Llpt = \frac{1}{2}Llp\phi = Lpd\phi$, th. $tp \parallel \phi d \parallel cv$.) fv : fc :: ca : ct . but (by 5) fv : fc :: ca : cv :: $(cv) ce$: ct , that is, $\frac{1}{2}t \mp x$: $\frac{1}{2}t$:: $\frac{1}{2}t$: $\frac{1}{2}t \pm a$ (which is also true, if t be the Diameter) :: $(\frac{1}{2}t + \frac{1}{2}t \mp x$: $\frac{1}{2}t + \frac{1}{2}t \pm a$::) $t \mp x$: $t \pm a$

:: $(\frac{1}{2}t \cup \frac{1}{2}t \mp x$: $\frac{1}{2}t \cup \frac{1}{2}t \pm a$::) x : a

7. Theref. $t \pm a$: a :: $t \mp x$: x . And,

8. $\frac{1}{2}t \mp x$: x :: $(\frac{1}{2}t : a$: $\frac{1}{2}t + \frac{1}{2}t \mp x$ or) $t \mp x$: $x + a$

9. Or, $\frac{1}{2}t + x$: $t \mp x$:: $\frac{1}{2}t$: $t \pm a$:: x : $x + a$. And,

10. $t \pm a$: $\frac{1}{2}t \pm a$:: $x + a$: a . Th. $u\Sigma$: ct :: ap : vs

11. Also, t : p :: $(t \mp x \times x)$: $\frac{1}{2}t \mp x \times x + a$: y^2

12. Th. $\frac{y}{x+a} = \frac{\frac{1}{2}t \mp x}{y} \times \frac{p}{t}$, or $\frac{ap}{at} = \frac{ca}{ap} \times \frac{p}{t}$

13. If vs , $u\Sigma$ ($\parallel ap$) meet ty in s , Σ ; then

$\left. \begin{array}{l} vs : ap :: (vt : at ::) cv : ua \text{ (by 8.)} \\ u\Sigma : ap :: (ut : at ::) cv : va \text{ (by 9.)} \end{array} \right\} \text{th.}$

$vs \times u\Sigma$: cv^2 :: ap^2 : vau :: p : t : $\frac{1}{4}tp$: $\frac{1}{4}t^2 = cv^2$.

Th.

$$Th. vs \times u\Sigma = \frac{1}{2}ap = \frac{1}{2}q \times q = cr \times ap.$$

Note, in the *Hyperbola*, supply the Fig. because here omitted to avoid Confusion; And observe the same in other places where 'tis required.

14. If from the Intersect. e, n of any Tangent ty , and a Circle descr. on vu , Perpendiculars eq, nf be erected; Then (bec. $\Delta s tvs, tu\Sigma$, Sim. $\Delta s tnp, tnf$.) $u\Sigma : nf :: eq : vs$,

$$Th. vs \times u\Sigma = eq \times nf = (\overline{v\phi \times \phi u})^{\frac{1}{2}} \times \overline{vf \times fu}^{\frac{1}{2}}) v\phi \times \phi u$$

or $vf \times fu = \frac{1}{2}q \times q (= \frac{1}{2}ap = cr \times ap)$ Th. ϕ and f are the Foci.

15. Also drawing $ps \perp pt$, or $\parallel qs$, and $cx, cl \perp po, pf$; then (bec. $\Delta s pps, spn$, Sim. $\Delta s fl\phi, sp\lambda$) fl (or t) $\times p\lambda = \phi l$ (or $2eo$) $\times fn = \frac{1}{2}p \times t$; Th. $p\lambda = p\chi = \frac{1}{2}p$.

16. Let $vs, u\Sigma$ (Perp. vu) intersect any Tangent tr in s, Σ , draw $sf, \Sigma f$: Then (bec. $vs \times u\Sigma = uf \times fv$ (by 14) th. $u\Sigma : uf :: fv : vs$, and the included $\angle s \Sigma uf, fvs$ are $\perp$, th. Δs are Sim. and $\angle f\Sigma u = \angle vfs$, but $\angle f\Sigma u + \angle uf \Sigma = \perp = \angle vfs + \angle uf \Sigma$, th.) $\angle sf \Sigma = \perp$. And $st = \phi t = fr = \Sigma r$. Th. if $s\Sigma$ be made the Diameter of a Circle, the Circumf. will pass thro' the Foci f, ϕ . (Supply the Figures.)

17. Draw $zt \parallel tp$, cutting cp in a ; Then $za = ar$.

$$\text{For (producing } rz \text{ to } n, \text{ and } cp \text{ to } d, x, \omega.) \frac{rz}{in} (= \frac{ap}{at} =$$

$$\frac{ca}{ap} \times \frac{p}{t}) = \frac{cv}{v\omega} \times \frac{p}{t}, \text{ th. } cv \times na \times zi = (v\omega \times zi^2$$

$$\times \frac{t}{p}) = v\omega \times viu \text{ (by 5.) th. } \overline{cv \times in \times iz} \text{ on } v\omega \times$$

$$ci^2 (= \overline{v\omega \times viu} \text{ on } \overline{v\omega \times ci^2} \text{ or } \frac{1}{2}x \times x \pm \frac{1}{2}x \times x)$$

$$\times v\omega) = v\omega \times cv^2, \text{ th. } v\omega \times cv = ci \times (\frac{ci \times v\omega}{cv}) \text{ on } in \times$$

$$iz, \text{ th. } \Delta cv\omega = \Delta ci x \text{ on } \Delta iz = (\text{by the like arguing}) \Delta cmd \text{ on } \Delta nmr, \text{ th. } \Delta azz = \text{and Sim. } \Delta ar d, \text{ Th. } za = ar;$$

which therefore are Ordinates to the Diameter pq . Therefore all Lines passing thro' the Centre are Diameters; And all Diameters pass thro' the Centre; And all Ordinates are parallel to the Tangent at the Vertex of their Diameters.

18. Hence

18. Hence (bec. $v\omega : ap :: vc : ac$) :: $tc : vc$ (by 6)
 Δ cpt (Δ cap $\frac{1}{2}$ tap) = Δ cv ω , th. Δ tvs = Δ sp ω ;
 And Fig. vix ω = Δ inz, also Fig. aptn = Δ axz, i. e.
 $ap \times an + pt \times ap = ax \times az = ap$ (σn) $\times an + pt$.

19. And $p\omega : ps (:: ax : az) :: 2pt : P$ = Parameter
 to the Diameter pq. For $P \times \frac{paq}{pq} (= \frac{az \times 2pt}{ax} \times \frac{paq}{pq} =$
 $\frac{az}{ax} \times paq \times (\frac{2pt}{pq} \text{ or } \frac{pt}{pc} \text{ or } \frac{\sigma t}{\sigma n}) \frac{\sigma t}{pa} = \frac{az}{ax} \times (aq \times \sigma t,$
 or $ac + cp \times \sigma t,$ or $tp + na \times (\sigma n) ap,$ or) $ax \times az =$
 $\frac{az}{ax} \times az \times ax) = az^2 = ar^2.$

Theref. the Squares of the Ordinates to any Diameter (D)
 are as the Rectangles under the Segments of that Diame-
 ter, i. e. $D \frac{1}{2} \times x \times x :: y^2 (:: D : P) :: D \frac{1}{2} \times X \times X . Y^2$
 And (bec. $\frac{1}{4}d^2 : \frac{1}{4}c^2 :: d : p$) c (the Conjug. Diam.) is = $\sqrt{dp}$.

20. Let $c\phi = r$, $v\phi = q$, $\phi f = d$, then

If $x = q$, ($t : p :: \frac{1}{4}tp (\frac{1}{2} + q \times q) : y^2$) $y = \frac{1}{2}p$

And $t : 2y :: \frac{1}{2} + q \times q : y^2$, Th. $y = \frac{1}{2} + q \times q \div \frac{1}{2}$

21. Since $\frac{1}{4}t^2 : (\frac{1}{4}pt = \frac{1}{2} + q \times q) = \frac{1}{4}(\frac{1}{2} + q)^2 \propto \frac{1}{4}t^2 ::$
 $\frac{1}{2}t : \frac{1}{2}p$. Th. $\frac{1}{4}t^2 : \frac{1}{4}(\frac{1}{2} + q)^2 :: \frac{1}{2}t : \frac{1}{2} + \frac{1}{2}p$, i. e.
 $\frac{1}{2}t : r :: 1 : \frac{1}{2} + \frac{1}{2}p$. Th. $\frac{1}{2}t : r :: r : \frac{1}{2} + \frac{1}{2}p$. Th.
 $r = \frac{1}{4}t^2 + \frac{1}{4}tp$; $q = \frac{1}{4}t^2 + \frac{1}{4}tp$ $\propto \frac{1}{2}t$; $d = \frac{1}{2} + tp$

22. Sup. any Ordinate mr, of an Ellipse, to meet the Focal
 Tangent in γ ; Then $\phi r = m\gamma$. Let $c\phi = r$, $cv = b$, cm
 $= n$, then $\phi p (\frac{v\phi u}{cv}) = \frac{b^2 - r^2}{b} = \frac{1}{2}p$, ct ($\frac{cv^2}{c\phi}) = \frac{b^2}{r}$

K k

th. ϕt

$$\text{tb. } \phi t = \frac{b^2 - r^2}{r}, \text{ and } m t = \frac{b^2 + r n}{r}, \text{ tb. } (b : r :: \frac{b^2 + r n}{r} :)$$

$$\frac{b^2 + r n}{b} = m \gamma; \text{ But (bec. } cv : \frac{1}{2} p :: v m u : m r^2, \text{ i. e. } b :$$

$$\frac{b^2 - r^2}{b} :: b^2 - n^2 :) m r^2 = \frac{b^4 - b^2 n^2 - r^2 b^2 + r^2 n^2}{b^2},$$

$$\text{and } \phi m^2 = r^2 - 2 r n + n^2, \text{ tb. } \phi r^2 = \frac{b^4 + r^2 n^2 + 2 r n b^2}{b^2};$$

$$\text{and } \phi r = \frac{b^2 + r n}{b} = m \gamma; \text{ 'Tis the same in the Hyperbola and Parabola.}$$

23. *Theref. if cb meet the Focal Tangent in τ , then $c\tau$ (ϕb) = cv = cu .*

$$24. (\text{Bec. } \frac{cv^2}{c\phi} (ct) : cv (c\tau) :: \frac{cv^2}{c\phi} + cv (\frac{ut}{vt}) : cv +$$

$$c\phi =) u\tau = u\phi, \text{ or } = vs = v\phi.$$

25. *The same being still supposed, (that is, that $x = q$) draw pu cutting any Ordinate mr in θ ; Then $\gamma\theta = \phi m$. (Supply the Figure.)*

26. *If on vu , a Circle be described, then bec.*

$$ap^2 : mr^2 : (:: vau : umv) :: a\omega^2, m\phi^2, \text{ tb.}$$

$$ap : mr :: a\omega : m\phi; ap : p\omega :: mr : r\phi,$$

$$\text{Or } ap : a\omega (:: mr : m\phi :: cb : cv) :: c : t.$$

$$\text{Tb. all (ap's : all a\omega's ::) Ellipse: } \odot t :: c : t :: ct : t^2$$

$$:: \odot \sqrt{tc} : (\odot \sqrt{t^2}) \odot t. \text{ Tb. Ellipse} = \odot \sqrt{tc}. \text{ And the}$$

$$\text{Ellipses } E : e :: TC : tc :: (\text{bec. } c = \sqrt{tp}) p^{\frac{1}{2}} t^{\frac{3}{2}} : p^{\frac{1}{2}} t^{\frac{3}{2}}$$

27. *Parallelograms circumscribing an Ellipse, having their Sides parallel to the Conjugate Diameters, are equal. For since $\Delta \mu p \gamma = \Delta \mu \tau b$ (by 18) and $L \text{ at } \mu =$, Tb. $\tau \mu : \mu p :: \mu \gamma : b \mu$, tb. $\tau p : \mu p :: b \gamma : \mu \gamma$ $\Delta c \tau p : \Delta c \mu p :: \Delta c \gamma b : \Delta c \mu b$, but $\Delta c \tau p = \Delta c \gamma b$ (by 18) tb. $\Delta c \mu p = \Delta c \mu b$, and $\Delta c \mu (\frac{1}{2} \square \text{ ft}) = \Delta \tau c \mu$, tb. $\Delta c b f = \Delta c p t$, tb. $\square c f = \square c t$; Also $\square c y (2 \Delta c \gamma y = 2 \Delta c \gamma b = 2 \Delta c \gamma \beta) = \square \beta c$, but*

$$\left\{ \begin{array}{l} \square y c : \square \pi c (:: y \gamma : \gamma \pi :: \pi p : p t) :: \square \pi c : \square c t \\ \square \beta c : \square k c (:: \beta v : v k :: k b : b f) :: \square k c : \square c f. \end{array} \right.$$

Tb.

Th. $\square \pi c = \square kc$, and theref. $4 \square \pi c = 4 \square kc$.

28. Th. if cp , cq be Semi-conjugate Diameters, and $p5$ $\perp$ cq , then $cq \times p5$ ($\square p9$) $= cv \times cb$.

Of the Hyperbolic Aſymptotes.

29. Since $ct \times ca = cv^2$ (by 6) if $ct = 0$, then $ca = \infty$, $vt = ut$, th. $vs = u\Sigma = \overline{vsXu\Sigma}^{\frac{1}{2}} = \frac{1}{4}tp^{\frac{1}{2}}$; Th. if vE or vA ($\perp vu$) be made $= \frac{1}{4}tp^{\frac{1}{2}}$, then cE , cA are called Aſymptotes to the Hyperbola or Oppoſite Hyperbolas.

30. $ca^2 : an^2 :: (cv^2 : vE^2 :: \frac{1}{4}t^2 : \frac{1}{4}tp :: t : p ::)$ $vau : ap^2 :: ca^2 - vau : an^2 - ap^2$; but $ca^2 > vau$, th. $an^2 > ap^2$; Th. n will always be without the Hyperbola, and conſeq. cn , tho' infinitely continued, ſhall never meet the Curve.

31. Alſo $cv^2 = ca^2 - vau$, th. $vE^2 (an^2 - ap^2) = npN$ or $n\pi N$; And $vE^2 = \gamma rK$ or γRK , th. $npN = \gamma rK$, but $pN < rK$, th. $pn > \gamma r$, Th. the Hyperbola and its Aſymptotes do continually approach nearer.

32. And if cL , cK are Aſymptotes, and by any point (p) of the Curve, a Tangent be drawn cutting cL , cK in 7 , 5 , Then $p5 = \frac{1}{4}p\omega \times P^{\frac{1}{2}} = p7$; $Lr \times r4 = 7p \times p5 = \frac{1}{4}p\omega \times P$; And $Lr = z4$.

33. Draw $r3$, parallel to any Semi-diameter pc , cutting the Aſymptotes in B , χ ; Then $Br\chi = cp^2$: For the Ordinates applied to the points p , a , meet the Aſymptotes in 7 , 5 , and

$$L, 4, \text{ th. } \frac{7p}{pc} \times \frac{p5}{pc} = \frac{Lr}{Br} \times \frac{r4}{r\chi}; \text{ Th. \&c.}$$

And (bec. $Br\chi = cp^2 = c\omega^2 = \chi^3 B$, alſo $Br = \beta\chi$) $r\chi = B\beta$.

34. If thro' any points P , ω , of the Hyperbola, be drawn right Lines $P3$, $\omega 2$, parallel to the Aſymptotes, Then (bec. $\Delta \omega \mu 2$. Sim. $\Delta P\gamma 3$, and $\omega \mu = P\gamma$, th. $\omega 2 = \gamma 3$, and $\mu 2 = P3$.) $P3 : \omega 2 :: (\mu 2 : \gamma 3 :: \mu c : c\gamma :: c\mu - P3 : c\gamma - \omega 2 ::)$ $c2 : c3$, Th. $P3 \times 3c = \omega 2 \times 2c$.

35. If the Segments cv , $c\delta$, $c3$, of the Aſymptote cy be $::$, Then ſhall $v\omega$, δs , $3P$ (drawn $\parallel$ to the other Aſymptote $c\mu$), be $::$.

For $v\omega : \delta s :: (c\delta : cv :: c_3 : c\delta ::) \delta s : 3P$.

But if $cv = ve = ed = \delta x$, &c. that is
if $cv, ce, c\delta, ck$, &c. be in a Contin. Aritb. } Proport.
Then $v\omega, e\lambda, \delta s, kb$, &c. are in a Contin. Harmon. }
And $e\lambda = \frac{1}{2}v\omega$, $\delta s = \frac{1}{3}v\omega$, $kb = \frac{1}{4}v\omega$, &c. as 'tis easie to demonstrate.

Let $es \parallel y\mu$, then $s\delta : es :: P_3 : yP$, and $s\delta : es :: v\omega : y\omega$. th. $s\delta^2 : es^2 :: P_3 \times v\omega : yP \times y\omega$, but $s\delta^2 = P_3 \times v\omega$ (by 35) th. $es^2 = yP \times y\omega$ or $yP\mu$, th. es is a Tangent, and $P\omega$ an Ordinate to the Diameter $c\delta$, th. $\Delta ySc = \Delta \mu Sc$, and Space $SsP = Ss\omega$, but $\Delta cs\delta = \Delta cs_9$, th. Space $s\delta_3P = s_9\omega$ but $\square c\omega = \square cs$, th. $\square \omega_9 = \square sv$. Th. Space, $s\delta_3P = s\delta v\omega$.

Likewise, if $ck, c\delta, ce$.., draw $kb, e\lambda \parallel c\mu$, then Space $e\lambda s\delta = \delta skb$, th. $bkv\omega = e\lambda P_3$, and bec. $v\omega \times P_3 = \delta s^2 = e\lambda \times kb$, th. $\frac{P_3}{e\lambda} = \frac{kb}{v\omega}$, Th. Asymptotic Spaces are as the Logarithms. So the Space atP_3 is the Log. of the Ratio's $\frac{P_3}{\tau a} = \frac{c a}{c_3}$; And the Log. of the Ratio of Equality $\frac{a\tau}{a\tau} (= 1)$ is 0.

In the Parabola.

36. Let $v\phi = q$, then $ag(\phi p) = x + q$, $\phi a = x - q$
and $y^2(p\phi^2 - a\phi^2 = x + q)^2 - (x - q)^2 = 4qx = px$
And $(4q)p$ is called the Parameter of the Axe.

37. Th. $v\phi = \frac{1}{4}p$; And $z\phi = 2\phi v = \frac{1}{2}p$.

38. Also $y^2 : r^2 (:: px : pX) :: x : X$.

39. Since $x < X$, th. $y < r$, th. the Curve of a Parabola runs off infinitely.

40. $p \times \overline{X - x} (pX - px) = \overline{r + y} \times \overline{r - y} (r^2 - y^2)$
that is, $px \Sigma \pi = r \Sigma x \Sigma k$.

41. Since $y^2 : r^2 (:: x : X) :: y : r + c$

Th. $\overline{r + c} \times y (= r^2) = Xp$.

42. Any

42. Any right Line nz cutting the Axe and Curve, will again if produc'd, cut the Curve in r , so that vi , vn , vm ::; Suppose it so, let rm , zi $\perp$ nm , then $vi : vn :: vn : vm :: vi + vn : vn + vm :: ni : nm$, and $iz^2 : mr^2 :: ni^2 : nm^2 :: vi^2 : vn^2 :: vi : vm$, th. the Point r is in the Curve.

43. Draw pt , so that $tv = va$, then pt shall touch the Curve in p ; since any other point (ϵ) in pt will be without the Curve: For (drawing $si \parallel pa$) $\frac{ie^2}{ap^2} = (\frac{it^2}{at^2} = \frac{\frac{1}{4}it^2}{va^2}$

$> \frac{ivt}{va^2} = \frac{iva}{va^2} = \frac{iv}{va} =) \frac{iz^2}{ap^2}$ Th. $it > iz$.

44. Th. ϕp (ga) = $t\phi$; and if $dp \parallel fv$, then $\angle dpy$ ($\angle \phi tp$) = $\angle \phi pt$.

45. If $ps \perp pt$, then $as = \frac{1}{2}p$; for $(pxav) \frac{1}{2}px \times 2av = (ap^2) 2av \times as$.

Th. if sn , $so \perp pd$, $p\phi$, then, (bec. Δs spx , $s po$, $s pa$ are = and Sim.) $px = po = sa = \frac{1}{2}p$.

46. And since $\phi p = \phi t$, and $\angle spt = \angle$, the Points s , p , t , are in a Semi-Circ. whose Centre is ϕ , Th. $\phi p = \phi s = \phi t$.

Hence the $\angle$ made by the Tangents, from the Extremities of any Ordinate, is $\frac{1}{2}$ that at the Focus made by Lines from the same Extremities.

Th. if the Ordinate passes thro' the Focus, the $\angle$ made by the Intersection of its Tangents will be a Right one.

And (bec. $av = vt$) $vs^2 (\frac{1}{4}ap^2) = \frac{1}{4}av \times p$.

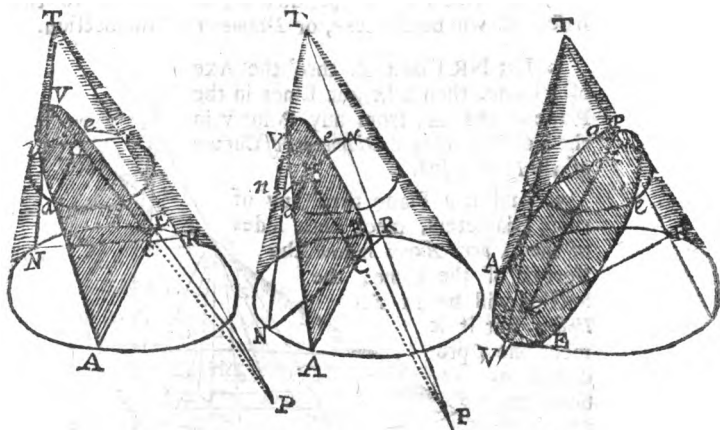
47. Let any Rt. Line zr , drawn $\parallel$ tp , cut the Curve in z , r , and lpd ($\parallel va$) in α ; let rm , zi ($\parallel pa$) cut pa in n , h . Then $ra = za$.

For Δapt ($\square wa$, bec. $2av = at$): $\Delta izn (::) ap^2 : iz^2 :: av : iv :: \square wa : \square wi$, th. $\Delta izn = \square ai$; and (by the like arguing) $\Delta mrn = \square om$, th. Fig. $izrm$ ($\Delta rnm - \Delta zrs = \square om - \square wi = \square hm$, th. $\Delta ran =$ and Sim. Δzah Th. $ra = za$, and pd is a Diameter, whose Ordinate is iz .

Hence, in the Parabola, every Line parallel to the Axe is a Diameter. And all Diameters are parallel the one to the other. Also all Ordinates are parallel to the Tangent at the Vertex of their Diameter. Theref. a Line, thro' the Vertex of the Diameter, drawn parallel to the Ordinate is a Tangent to the Parabola.

48. And

ae) the Rectangles of the intercepted segments of that Line will be, as the Rectangles of the Segments of the respective Parallels: For

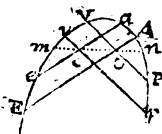


Let the right Line VP be the Common Section of the ΔNTR , (passing thro' the Vertex of the Cone, but not thro' the Axe) and the Section AVE.

Then $RC \times CN = EC \times CA$, and $rc \times cn = ec \times ca$ (by 64. 23.) Also $rc:RC::cP:CP$, and $nc:NC::Vc:VC$.

Th. $rc \times cn:RC \times CN::Pc \times cV:PC \times CV$.
Or $ec \times ca:EC \times CA::Pc \times cV:PC \times CV$.

3. Th. in any Conic Section, if two Parallels are cut by two others, and all terminate at the Curve, the Rectangles of the Segments shall be Proportional, i. e. $ace:ACE::(mcn:mcn::)pcv:PCV$.

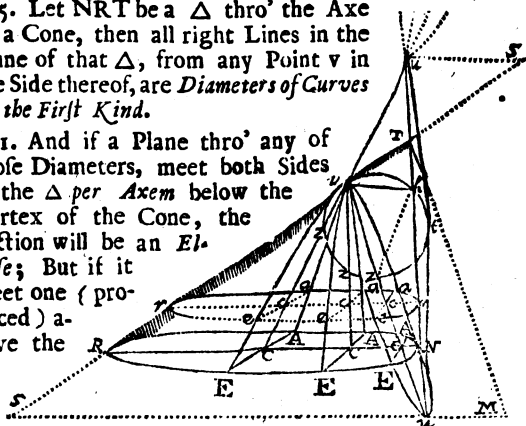


4. Let any Plane AVE cut the Base of a Cone in A, E, then NR drawn Perp. to AE, and thro' the Centre, bisects AE: Let a Plane NTR cut the Cone thro' NR and the Axe; and a Plane (nare) cut it || to the Base, then the com. Sect. ae, and nr, of the Planes nar, ave, and nar, nTr are || to EA and NR; but nr passes thro' the Centre, th: bisects ea in c; also vC (the com.

com. Sect. of the Plane AvE and that *per Axem*) bisects all right Lines drawn $||$ to AE in the Section, th. vc is a *Diameter* of the Section, and AC, ac , are *Ordinates*. If the Plane *per Axem* be Perpendicular, or Oblique to the Base, vC will be the *Axe*, or *Diameter* of the Section.

5. Let NRT be a Δ thro' the *Axe* of a Cone, then all right Lines in the Plane of that Δ , from any Point v in the Side thereof, are *Diameters of Curves of the First Kind*.

1. And if a Plane thro' any of those *Diameters*, meet both Sides of the Δ *per Axem* below the Vertex of the Cone, the Section will be an *Ellipse*; But if it meet one (produced) above the



Vertex, the Section will be an *Hyperbola*, and the Sections of the opposite Cones will be equal *Hyperbolas*. For

1. $vc : vC :: rc : RC$, and $uc : uC :: nc : NC$
Th. $vcu : vCu :: (ncr : NCR ::) ac^2 : AC^2$

2. Draw $TM, vh || vu, NR$; make $vu : vh :: us : P$ the *Parameter*; For $vc : cr :: vu : vs :: TM : Ms$, and $uc : cn :: vu : vh :: TM : Mu$, th. $vcu : (ncr) ca^2 :: (vu^2 : us \times vh (vu \times P) ::) vu : P$. Also $TM^2 : sMu :: vu : P$.

The same appears more general by 69, 2. For (if D, d , be the *Transverse* and *Conjugate Diameters*; y, Y , *Ordinates* to the *Diameter* D ; x, X , their *Abscissæ*) $xx \overline{D+x} : X \overline{D+x} :: y^2 : Y^2$. And if an *Ordinate* (r) be applied to the Centre, 'twill become $= \frac{1}{2}d$, then $\frac{1}{4}D^2 : \frac{1}{4}d^2 :: xx \overline{D+x} : y^2$, Also, let (1) $D : d :: d : P$, And (2) $D : x :: P : r$, then (bec. $\overline{D+x} : x :: P+r : r$ (by 2) th. $r \times \overline{D+x} = x \times \overline{P+r}$, but $D : P :: D^2 : d^2$ (by 1) :: $x \times \overline{D+x} : y^2$ (by)

(by 69.2) :: $x : y$ (by 2) :: $x \times D \mp x : x \times D \mp x$) $x \times P \mp x$
 $= y^2 = DPx \mp Px^2 \div D$. And bec. $DP = d^2$, $dp = D^2$,
 th. $Pp = Dd$, and $\frac{1}{4}d^2$ ($\frac{1}{4}DP$) $\mp \frac{1}{2}D \times \frac{1}{2}P$.

2. If the Plane be Parallel to one Side of the Δ per
 Axem, the Section will be a Parabola. For

1. $vc : vC :: (cr : CR :: ncr : NCR ::) ca^2 : CA^2$.

2. Make $NTR : NR^2 :: Tv : P$, the Parameter; For
 $NC(vh) : vT :: NR : RT$, and $RC : Cv :: RN : NT$,
 th. $CA^2 (NCR) : TvC :: (NR^2 : NTR :: P : Tv ::)$
 $PxvC : TvC$, th. $CA^2 = PxvC$. Or by 69. 2. (bec.
 $D-x = D-X$, since $D = \infty$) $x : X :: y^2 : r^2$.

And if $x : y :: y : P$, then $r^2 = Px$ every where; For
 $Px(y^2) : r^2 :: (x : X ::) Px : PX$.

6. If thro' v and h a Circle be describ'd, whose Segment vzh
 contains an $L = L Tv h$, the Chords vz shall be the Parameters
 to the Diameters vu , vc , &c. For in the Ellipse and Hy-
 perbola, (bec. Δvhz Sim. Δvsu th.) $vuxP (v h \times su) =$
 $vuxvz$, th. $P = vz$. And in the Parabola (bec. Δvhz
 Sim. ΔvRC) $vCxP (vh (NC) \times CR) = vCx vz$, th.
 $P = vz$.

And ut Tangents to the Circle from the Extremities of
 the Diameters, shall be the Distance of the Foci; For ut²
 $= vu^2 \mp vz \times vu$ (by 64. 19.) = square of the Distance of
 the Foci (by 68. 22.)

7. In any Conic Section $vp\pi$, where ap , $a\pi$ are Ordi-
 nates, va the Diameter, vv a Tangent; Let pc , πv ($\parallel va$)

be infinitely small, Then (since $ap^2 = p \div t \times t \times va \mp va^2$
 $=$ (bec. $va : t :: 1 : \infty$) $p \div t \times t \times va$, and so

$a\pi^2 = p \div t \times t \times va$, th. $ap^2 : a\pi^2 :: va : va$,
 or) $vc^2 : vv^2 : pc : \pi v$. Th. $cp : cv :: (va : ap ::$
 $ap : tp \div t ::) 1 : \infty$. And (bec. $vp^2 = (vc^2$
 $\mp cp^2 =) vc^2$) $vp = vc$. Th. Chord $vp = Tan-$
 gent $vc = Arc vp$; And $pc : \pi v :: vp^2 : v\pi^2$.



8. Let any Line vf cut pc , πv , produced, in e , f , then
 $cp : ce :: 1 : \infty$, th $ec = ep$, and $\Delta vce = \Delta vef$. Th.
 $\Delta vpe : \Delta v\pi f :: (\Delta vce : \Delta vef :: vc^2 : vv^2 ::) ve^2 : vf^2$.

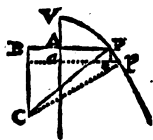
70. Curve Lines are by some distinguish'd thus; That,
 whose Nature is express'd by an Equation, wherein the

two Indeterminate Quantities represent Right Lines, or one of them a Curve Line, they call an *Algebraic*, or *Transcendent Curve*: And those, the Exponent of whose Equation are variable Quantities, they call *Exponential Curves*.

But since the Relations between the Indeterminate Quantities in the Equation of a Curve may be infinitely varied, therof. the Number of such Curves is infinite; Some of which, besides those of the *First* and *Second Kind*, *Geometers* have consider'd; But of these, we shall in this place, only take Notice of the Primary Properties of the most Simple *Transcendent*, and *Exponential Curve*; the *Cycloid*, and *Logarithmic Line*; their Use being very considerable.

71. If the Tangents to the Points of any Curve be produced till they become equal to their respective Parts of the Curve, their Extremities will be in another Curve; which is said to be describ'd by *Evolving* the former, the Curve Evolv'd is called the *Evoluta*, and the Tangents so prolong'd are called the *Radii of the Evoluta*, or *Radii of the Curvature*; Which therefore are all perpendicular to the Curve describ'd by the Evolution: So that the Points of the *Evoluta* are but the Intersections of Perpendiculars, to the Curve describ'd, that are infinitely near the one to the other.

Th. sup. the Ordinates AP, ap, infinitely near, and C the Intersection of the Perpendiculars to the Curve Vpp, in P and p. Draw CB || AV,



then $Pe : Pp :: PB : PC$, i: e $\dot{x} :$

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}} \Big|^{1/2} :: z : \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}} \Big|^{1/2} \times z \div \dot{x},$$

But whilst x and y increase by $\dot{x}$ and $\dot{y}$, PC becomes pC, th. does not increase, Conseq. the *Flux.* of (PC)

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}} \Big|^{1/2} \times z \text{ is } \frac{\dot{z} \dot{x}^2 + \dot{z} \dot{y}^2 + z \dot{y} \ddot{y}}{\dot{x}^2 + \dot{y}^2 \Big|^{1/2} \times \dot{x}} = 0, \text{ th. (bec.}$$

$$\dot{y} = \dot{z}) z = \frac{\dot{x}^2 + \dot{y}^2}{-\dot{y}}, \text{ but } Pe : Pp :: PB : PC, \text{ i: e } \dot{x} : \frac{\dot{x}^2 + \dot{y}^2}{\dot{y}} \Big|^{1/2}$$

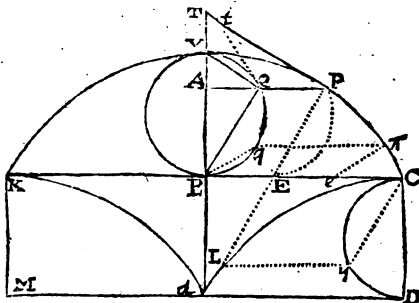
$$\frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{|\dot{x}^2 + \dot{y}^2|^{\frac{1}{2}}} :: \frac{\frac{\dot{x}^2 + \dot{y}^2}{-\dot{y}}}{-\dot{x}} : \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\dot{y}} \text{ the Ray of the}$$

Evoluta or *Radius of the Concavity* of any Curve, or the Radius of a Circle that touches the Curve in P or p.

72. The Curve Line describ'd by the Motion of any Point of the Circumf. of a Circle revolving on a Right, or Circular Base is called a *Cycloid*. Thus (when the Base is a Right Line) the Arcs EP, BQV, are those of the generating Circle, in their due places, whilst the Points P, V of the *Cycloid* are describ'd: So that (drawing PQ || CB) CE = arc EP = arc BQ, th. arc QV (BE) = QP.

Whence, to describe this Curve, make $d : c :: BV : BC = \text{Arc } BQV$, divide BC , and the arc BQV into the like Number of $=$ parts, in the points E, e, F, f , and Q, q , &c. draw BQ, Bq , complete the Pgrs. $BQPE, Bq\pi e$, the Points P, π , will be in the Curve.

1. And where a part of some Curve $VQ = x$, and PQ



$\therefore y$, then the Tangent $Qt = yx \div \dot{y}$; But if $x =$ Arc of a Circle, and $x : y :: m : n$, th. $x = ym \div n$, th. $Qt = ym \div n = x$. But in the *Primary, Protracted, or Contracted Cycloid*, $n = , > , < m$, Th. in the *Primary Cycloid* $Qt = y = x$. Th. $\angle tPQ (\frac{1}{2} \angle tQA) = \angle VQA$, Th. the Tangent Pt to any Point P of the Cycloid is $\parallel$ to the respective Chord VQ .

2. Th. QA: AV::PA: AT; And PT : TA :: QV
($\sqrt{dx}\sqrt{v}$): VA ($\sqrt{v}\times\sqrt{v}$): $\sqrt{d}$: $\sqrt{v}$.

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3. And

3. And (bec. $VQ = \sqrt{dv}$, and $\parallel PT$) $v : \sqrt{dv} :: \dot{v} : d^{\frac{1}{2}} \dot{v} - \frac{1}{2} \dot{v} = \dot{z}$, th. the *Cycloidal Arc* $VP = (2\sqrt{dv}) 2 VQ$,

i. e. The Curve is = twice the corresponding Chord.

4. If the Semicycloid CLd be Evolv'd, the Curve describ'd will be a Semicycloid, Similar and Equal to the Evoluta. For the Ray of the Curvature PL is a Tangent to the Cycloid in L, th. $\frac{1}{2} LP = \frac{1}{2} \text{arc } LC = Cy = LB$, th. $EP = Cy (= BQ)$ but $EP \parallel Cy$, th. $(VB = \text{and } \parallel CD)$ a Circle thro' B, Q, V, is = Circle DyC, and the arc $BQ = \text{arc } Cy = yL = CE$. Th. the Curve CP is a Cycloid, equal and like to CLd.

Th. if CDdk M be two like Cycloidal Plates, a Weight suspended at d, by the Chord dLP, when put in Motion, will describe a Cycloid, whose Axe is $\frac{1}{2}$ the Length of the describing Pendulum.

These things will also appear, by finding the Value of the Ray of the Evoluta, according to the general Rule (Art: 71.)

$$\begin{aligned} \text{Since } y &= a + s, \text{ th. } \dot{y} = r\dot{v} \div s + r\dot{v} - v\dot{v} \div \\ s &= \frac{2r\dot{v} - v\dot{v}}{s} \div s = \dot{v} \times \frac{2r - v}{s} \div \dot{v}^{\frac{1}{2}}, \text{ th. } \ddot{y} = - \\ r\dot{v}^2 \div vs, \text{ th. PL (or } \frac{\dot{v}^2 + \dot{y}^2}{-\dot{v}\dot{y}} \Big|)^{\frac{3}{2}} &= \frac{8r^3 \dot{v}^6 \div v^3 \Big|^{\frac{1}{2}}}{-\dot{v}\dot{y}} = \\ \frac{8r^3 \dot{v}^6 \div v^3 \Big|^{\frac{1}{2}} \times vs}{r\dot{v}^3} &= \frac{16r^2 - 8rv}{\dot{v}^{\frac{1}{2}}} = 2 \times \frac{4r^2 - 2rv}{\dot{v}^{\frac{1}{2}}} \\ &= 2 \times \frac{AQ^2 + AB^2}{\dot{v}^{\frac{1}{2}}} = 2QB = 2PE. \end{aligned}$$

Th. if $v = 0$, $PL = Vd = (2 \times 4r^2)^{\frac{1}{2}} 4r = 2d$. And (since LP is Perp. to the Curve and $\parallel BQ$) a Line PT $\parallel VQ$ shall be a Tangent. Also any part (CL) of the Cycloidal Arc, is = to twice (Cy) the corresponding Chord.

The Cycloidal Space (VCB) between the Curve and the Circle is = (bec. its Elements are = to their corresponding Arcs, which are an *arith. Progr.*) $\frac{1}{2} \square CBV = \frac{1}{2} \text{er} = \text{Generating Circle}$.

73. Let two right Lines move at right $\angle$ s, the one thro' equal Spaces in equal Times, the other thro' Spaces decreas.

2. $\text{Sec. MP} = \text{Secant of } 45^\circ \text{ to the Radius BM and Tangent BP, Th. the Logarithmic Line is composed of all the Secants.}$

4. Since $\dot{x} = \dot{y}$, th. Flux. of the Area ($\dot{xy}$) $= \dot{xy}$. Th. the Infinite Space $ACec = sy = \dot{xy} \cdot ACT = 2\Delta ACT$. Th. the Space between any two Ordinates Y, y , is $= s \times Y - y$.

1. The Arcs Ar, $r\theta$, &c. or the L s A Ca, a Ca, &c. being supposed infinitely small, and in an *Arith. Progression*; and the Radii, or Sides about those L s in a *Geometric Progression*, Th- the L s made by the Radii and Curve are every where equal. 2. If

2. If

1. Where the Ordinates are parallel; $\dot{\lambda} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$,
and $\dot{a} = \dot{x}y$.

2. Where the Ordinates respect a Point; (Let $r = \text{Rad.}$
of a Circle describ'd on that Point, $x = \text{Abscissa}$, and
 $y = \text{that part intercepted between the Point and}$

Curve) $\dot{\lambda} = y^2 \dot{x}^2 + r^2 \dot{y}^2^{\frac{1}{2}} \div r$, and $\dot{a} = y^2 \dot{x} \div$
2r. And by substituting in the room of $\dot{x}^2$ or $\dot{y}^2$, or of

$\dot{x}$ or y , their Values from the Equation of the Curve,
there will be produced an Equation whose Fluent is the
Length, or Area sought.

Thus in Case 1. If $y = x^n$, then $\dot{a} = \dot{x}x^n$, th. $a =$
 $(\frac{x^{n+1}}{n+1} = \frac{xx^n}{n+1} =) \frac{1}{n+1} xy$. And the *Asymptotic Space*,
in the Hyperbola, (n being there *Negative*) is $=$
 $\frac{1}{-n+1} xy$; Therefore if $n <, =, > 1$, the Space will

be *Finite, Infinite, or More than Infinite*.

More Examples are given where Occasion requires, th.
are here, for Brevity's sake, omitted.

There are various other ways of finding the *Lengths*,
or *Areas* of particular *Curve Lines*, or *Planes*, which
may very much facilitate the Practice; as for Instance,
in the Circle, the Diameter is to Circumference as 1 to

$$\frac{16}{5} - \frac{4}{239} - \frac{1}{3} \frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{3} \frac{16}{5^5} - \frac{4}{239^5} - \text{Ec.} =$$

3.14159, Ec. = π . This Series (among others for the
same purpose, and drawn from the same Principle) I re-
ceiv'd from the Excellent Analyst, and my much E-
steem'd Friend Mr. John Machin; and by means there-
of, Van Ceulen's Number, or that in Art. 64. 38. may
be Examined with all desirable Ease and Dispatch.

Whence in the Circle, any one of these three, a, c, d ,
being given, the other two are found, as, $d = c \div \pi$
 $= \frac{a \div \frac{1}{4}\pi^{\frac{1}{2}}}{c}^{\frac{1}{2}}$, $c = d \times \pi = \frac{a \times 4\pi^{\frac{1}{2}}}{a}$, $a = \frac{1}{4}\pi \times d^2 =$
 $c^2 \div 4\pi$ And

And any *Segment of a Circle* = *Sector* — *Triangle* there-
in = $\frac{1}{2}ar - \frac{1}{2}sr$; But like Segments are as their respec-
tive Circles, i. e. π (the Circle where $r = 1$): $\frac{1}{2}a - \frac{1}{2}s$

$\therefore a$ or $\frac{1}{4}\pi^2 d$ (the Circle proposed) : $a - sa$ $\div$
 2π , or $a - s \times d^2 \times \frac{1}{4}$; but a (of n degrees) in parts of 8 is
 $= \frac{1}{180} \pi n$, th. Segment = $\frac{1}{180} \pi n - sa \div 2\pi$, or $\frac{1}{180} \pi n - s$
 $\times d^2 \times \frac{1}{4}$.

Or let v = *Versed Sine* or *Height of the Segment*, and
 s = *Sine of n the double Co-Arc* (in degr. and parts) to
the *Sine* 1 — $2v \div d$ (from the Table of *Nat. Sines*), then

$\frac{1}{180} \pi n - s \times a \div 2\pi$ = *Segment*.

78. To find the *Surfaces* (s) of Solids generated by
the Rotation of Planes about an Axis.

1. Where the Axis of Rotation is the *Abscissa*, $\dot{s} =$
 $\dot{y}^2 + \dot{x}^2 \Big| \frac{1}{2} \times c \div r \times y$. Examples

In a *Cone*, $y = \frac{rx}{b}$, th. $\dot{s} = (\frac{c}{r} \times \frac{rx}{b} \times \sqrt{\frac{r^2 \dot{x}^2}{b^2} + \dot{x}^2}$
 $= \frac{c \dot{x} x}{b} \times \sqrt{\frac{r^2 + b^2}{b^2}} = \frac{c \dot{x} x}{b} \times \frac{\sqrt{r^2 + b^2}}{b} = \frac{c \dot{x} x}{b} \times$
 $\frac{l}{b}$, th. $s = (\frac{c x^2}{2b^2} l) \frac{1}{2} cl$ or $\frac{1}{2} \pi d l$ (when $x = b$).

Th. $\frac{1}{2} \pi d l : \frac{1}{2} \pi d r :: l : r$. And if the Side of the *Fru-*
stum of a *right Cone* = b , its *Curv'd Surface* will be =
 $C + c x \frac{1}{2} b$. In a *Sphere*,

$y = \frac{r^2 x^2 - x^2}{2rx - xx} \Big| \frac{1}{2}$, th. $\dot{y}^2 = \frac{r^2 \dot{x}^2 - 2rx \dot{x}^2 + x^2 \dot{x}^2}{2rx - xx}$
th. $\dot{y}^2 + \dot{x}^2 \Big| \frac{1}{2} = \sqrt{\frac{r^2 \dot{x}^2}{2rx - xx}}$ th. $\dot{s} = (\frac{c}{r} \times \frac{r^2 x^2 - x^2}{2rx - xx} \Big| \frac{1}{2})$

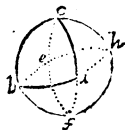
$\times r \dot{x} \times \frac{r^2 x^2 - x^2}{2rx - xx} \Big| \frac{1}{2} c \dot{x}$. Th. $s = cx (\pi dx$ the *Surf. of the*
Segment) = (when $x = d$) cd or πd^2 the *Surf. of the*
Sphere.

Th.

Th. $\pi d x : \pi d^2 :: x : d$. And $\pi d X - \pi d x : \pi d^2 :: X - x : d$. Hence the Curved Surfaces of Segments, or Frustums of Spheres, cut by parallel Planes, are equal to the corresponding Surfaces of the Sphere's Circumſcr. Cylinder.

And $\frac{1}{2} c d : c d :: \frac{1}{2} c : c$, i: e. The Surface of the Sphere contain'd between any two great Circles, is to the whole Surface, as the $\angle$ of Inclination, is to 4 $\angle$ s.

Th. 1: $r :: 3 Ls$ (of any Sph. Δabc) — 2 $\angle$ s (in parts of r): Surf. of the Sph. Δ . For $\Delta bac = feh$, and $\Delta bec = fah$; And 4 $\angle$: $\angle cba + bca + bac :: \epsilon$ (Surf. Sph.): $bahcb + cbfac + abeca$, i: e, 4 $\angle$: 3 $Ls :: \epsilon : 2\Delta + \frac{1}{2}\epsilon$, or 2 $\angle$: 3 $Ls :: \epsilon : 4\Delta + \epsilon$, th. 2 $\angle$ ($\frac{1}{2}\epsilon$): 3 $Ls - 2\angle$ (δ): $\epsilon : 4\Delta$, th. $2\epsilon\Delta$ ($\delta\epsilon$) = $2rc\delta$, Th. $\Delta = r\delta$, th. $\epsilon\epsilon$.



2. Where the Axis of Rotation is parallel to the Absciſſa, (let the longest Perp. from the Curve to the Axis of Ro-

tation be p) $\epsilon = \overline{p - y \times \dot{y}^2 + \dot{x}^2}^{\frac{1}{2}} \times c \div r$.

3. Where the Axis of Rotation is the Base of the Curve,

$\epsilon = \overline{p - x \times \dot{y}^2 + \dot{x}^2}^{\frac{1}{2}} \times c \div r$.

4. Where the Axis of Rotation is a Tangent to the Vertex

of the Curve, $\epsilon = \overline{\dot{y}^2 + \dot{x}^2}^{\frac{1}{2}} \times x \times c \div r$.

79. To find the Contents (χ) of Solids generated by the Rotation of Planes about an Axis.

1. Where the Axis of Rotation is the Absciſſa, and the generating Plane adjacent to it, or to the Tangent at the

Vertex. (1) $\chi = \dot{x} y^2 \times c \div d$, or (2) $\chi = \dot{x} p^2 - y^2 \times c \div d$.

Examples in Case 1. of this.

In a Cone, $y = \frac{rx}{b}$, th. $y^2 = \frac{r^2 x^2}{b^2}$, th. $\chi = \frac{cr^2 x^2}{db^2}$, th.

$\chi = (\frac{cr^2 x^3}{3db^2} = \frac{1}{2} crb =) \frac{1}{12} \pi d^2 b$; Theref. Cylinder

($\frac{1}{4} \pi d^2 b$): Cone ($\frac{1}{12} \pi d^2 b$): $3 : 1$.

Or because the Elements of a Pyramid, or Cone, are as the Sq. Numbers from 0; th. their Sum, i: e. the Pyra. mid, or Cone, is $\frac{1}{3}$ of ſo many times the greateſt, i: e. $\frac{1}{3}$

M m

of

of the Circumscr. *Prism*, or *Cylinder*. Or sup. B the Base of a $\triangle$, b, b , &c. its Parallels, $B + b = z$, or d , Th. $b = B - d$, and $b^2 = B^2 - 2Bd + d^2$, th.

all the b^2 's $= bB^2 - bBd + \frac{1}{3}bd^2 = b \times \overline{B^2 - Bd + \frac{1}{3}d^2}$
 $= b \times \overline{Bb + \frac{1}{3}d^2} = \frac{1}{3}b \times 3Bb + d^2 (B - b)^2 = \frac{1}{3}b \times$
 $\overline{B^2 + Bb + b^2} = \frac{1}{3}b \times \overline{z^2 - Bb} = \text{Frustrum of a sq. Pyramid.}$
 Th. the Frustr. of any *Pyramid* or *Cone*, (A, a being the

Areas of the Bases) is $b \times \overline{A + \sqrt{Aa} + a} \times \frac{1}{3}$. If the Bases are Circles, D, d , their Diameters, then the Frustr. $=$
 $b \times \overline{z^2 - Dd} \times \frac{1}{12}\pi$, and the whole *Cone* $= \frac{1}{12}\pi D^2 b$, as before.

If the Frustr. of a *Cone* be cut by a Diagonal Plane passing thro' the op. Extremities of the Bases, then the

Greatest, or *Least Frustr.* $= b \times \overline{\frac{1}{2}Dd + D^2}$ or $d^2 \times \frac{1}{12}\pi$; This is evident, as also, the preceding *Theorems*, upon Sight of a Frustrum of a *Right Pyramid* of Four (or any even Number of Sides) reduced into its Component *Pyramids* and *Prisms*.

In the *Sphere*, $y^2 = dx - x^2$, th. $\chi (dcxx - cx^2 \div d)$
 $= cxx - cx^2 \div d$, th. $\chi = \frac{1}{2}cx^2 - cx^3 \div 3d = \frac{1}{2}\pi dx^2 - \frac{1}{3}\pi x^3 =$ (when $x = d$) $\frac{1}{6}cd^2$ or $\frac{1}{6}\pi d^3 = \frac{1}{6}d \times \text{Surf.}$
 (πd^2), Th. *Sph.* ($\frac{1}{6}\pi d^3$) : *Cylind.* ($\frac{1}{4}\pi d^3$) :: 2 : 3. Or
 since the *Ring* $DE (= \odot CE - \odot AE = \odot CA) = \odot AN$, all the *Rings* DE = all the $\odot AN$ s, th. *Figure* $YEXZY = \text{Cone } CXZ$
 $= \frac{1}{3} \text{Cylind. And Hemisph. (Cyl. - Cone)} = \frac{2}{3} \text{Cylinder.}$ Th. *Cone*, *Sph.* and *Cylind.* are as 1, 2, and 3.

Or since $s^2 = c^2 (dv) - v^2$, th. all the s^2 's $= \frac{1}{2}v \times c^2 - v \times \frac{1}{3}v^2 (= \frac{1}{2}v \times s^2 + v^2 (c^2) - \frac{1}{2}v \times \frac{2}{3}v^2) =$
 $\frac{1}{2}v \times s^2 + \frac{1}{3}v^2$ or $\frac{1}{2}v \times \odot s + \frac{1}{3}\odot v$, or $v \times 2s^2 + \frac{4}{3}v^2 \times \frac{1}{4}\pi$,
 or $v \times 3 \times 2s^2 + 4v^2 \times \frac{1}{4}\pi = \text{Segment of a Sphere. Th.}$
 $\frac{1}{2} \text{Sph.} = \frac{1}{6}\pi d^2 r$, and *Sph.* $= \frac{1}{6}\pi d^3$, as before.

And

And, if a *Sphere* be cut by any two Parallel Planes, S, s, shall be the Radii of the *Fruustum's* Bases, its Height (V—v) call *b*, then $V = b + v$: but bec. any Segment of a Sph. $= \pi r v^2 - \frac{1}{3} \pi v^3$, Th. Fruft. $= \overline{2rv - v^2} \times \pi b + \overline{r - v} \times \pi b^2 - \frac{1}{3} \pi b^3 = (\text{bec. } 2rv - v^2 = s^2, \text{ and } r - v = S^2 - s^2 + b^2 \div 2b, \text{ by 65.23}) b \times S^2 + s^2 + \frac{1}{3} b^2 \times \frac{1}{2} \pi$.

Th. $v \times S^2 + \frac{1}{3} v^3 \times \frac{1}{2} \pi = \text{Segment}$, and $\frac{1}{6} \pi d^2 r = \frac{1}{2} \text{Sphere}$, as before.

In an *Oblong Spheroid*, let $a = \frac{1}{2} t = \frac{1}{2}$ the *Transverse*, $r = \frac{1}{2}$ the *Conjugate Axe*, then (bec. $a^2 : r^2 :: 2ax - x^2 :$

$$y^2 = \frac{2axr^2 - x^2r^2}{a^2}) \chi = \frac{2axr^2xc - x^2xr^2c}{a^2d}, \text{ th. } \chi =$$

$$\frac{r^2ax^2c - \frac{1}{3}r^2x^3c}{a^2d} = \text{Segment}, \text{ th. } \frac{2r^2ac}{3d} = \frac{1}{6}d^2\pi a = \frac{1}{2}$$

Spheroid, and $\frac{1}{6}d^2\pi t = \text{Spheroid}$. Th. *Cylind.* $(\frac{1}{4}\pi d^2t) : \text{Spheroid} (\frac{1}{6}\pi d^2t) :: 3 : 2$. And *Cone* $(\frac{1}{3}\text{Cylinder}) = \frac{1}{2} \text{Spheroid}$.

Or (sup. *Parameter* $= p$, and $p \div t = r$) since $y^2 (r \times \overline{2ax - x^2}) = r \times \overline{a^2 - a - x}^2$ or $ra^2 - rm^2$, th. all the $y^2s = b r \times \overline{a^2 - \frac{1}{3}m^2} = \frac{1}{3} b r \times \overline{3a^2 - m^2} = \frac{1}{3} b r \times \overline{2ra^2 + ra^2 - rm^2} = (\text{bec. } y^2 = ra^2 - rm^2, \text{ and } r^2 = ra^2) \frac{1}{3} b \times \overline{2r^2 + y^2}$, Th. $\frac{1}{3} b \times \overline{2 \odot r + \odot y}$ or (if *D, d* be the *Diameters* of the *Bases*) $b \times \overline{2D^2 + d^2} \times \frac{1}{2} \pi = \text{Fruft. of a Spheroid cut by two parallel Planes, the one passing thro' the Centre}$. Th. the *Spheroid* $= D^2t \times \frac{1}{6} \pi$, as before.

$$\text{If } y = x^2, \text{ then } \chi = \frac{c}{d} x x^{2n}, \text{ and } \chi = \frac{c x^{2n+1}}{2n+1 \times d}$$

Th. if $m = \frac{1}{2}$, $\chi (= cx^2 \div 2d = cy^2x \div 4r) = \frac{1}{4} crb = d^2b \times \frac{1}{8} \pi = \text{Obtuse Parabolic Conoid}$, Th. *Conoid* $(\frac{1}{8}\pi d^2b) : \text{Cylind. } (\frac{1}{4}\pi d^2b) :: 1 : 2$. And *Cone, Conoid, and Cylinder* are as 2, 3, and 6.

2. Where the Axis of Rotation is the Base (b) of the Curve, and the Plane adjacent to it, or to the Tangent at the Vertex (1.) $\dot{\chi} = \dot{x} \overline{py - xy} \times c \div r$, or (2.) $\dot{\chi} = \dot{x} \times \overline{p - x} \times \overline{b - y} \times c \div r$.

Example in Case 1. If $y = x^n$, $\dot{\chi} = px^{n+1} - \dot{x}x^{n+1} \times \frac{c}{r}$, th. $\chi = \frac{cp x^{n+1}}{r \times n + 1} - \frac{cx^{n+2}}{r \times n + 2}$. Th. if $n = \frac{1}{2}$, as in the common Acute Parabolic Conoid, then $\chi (= \frac{cx^2 y}{\frac{1}{2}r} - \frac{cx^2 y}{\frac{1}{2}r} = \frac{1}{2}\pi d^2 y - \frac{1}{2}\pi d^2 y) = \frac{2}{15}\pi d^2 b$. Th. is to the Cylinder as 8 to 15.

Or in this Solid, call the Axe of the Curve B, its Parallels b, b, &c. and $B - b = \delta$. Th. $b = B - \delta$, and $b^2 = B^2 - 2B\delta + \delta^2$, th. all the b^2 s = $bB^2 - \frac{1}{3}b \times 2B\delta + \frac{1}{3}b\delta^2 = b \times B^2 - \frac{2}{3}B\delta + \frac{1}{3}\delta^2 = \frac{1}{3}b \times 2B^2 + \frac{1}{3}\delta^2 + (B^2 - 2B\delta)b^2 - \delta^2 = \frac{1}{3}b \times 2B^2 + b^2 - \frac{2}{3}\delta^2$ or $\frac{1}{3}b \times 2 \odot B + \odot b - \frac{2}{3} \odot \delta =$ Fruft. of an Acute Parabolic Conoid, = (if D, d be the Diameters of the Bases) $2D^2 + d^2 - \frac{2}{3}\delta^2 \times b \times \frac{1}{2}\pi$.

3. Where the Axis of Rotation is a Tangent at the Vertex of the Curve, and the Plane adjacent to it, or to the Abscissa. (1.) $\dot{\chi} = \dot{y} x^2 \times c \div d$, or (2.) $\dot{\chi} = xy \dot{x} \times c \div r$. So that (in Case 2.) if $y = x^n$, $\dot{\chi} = \frac{c}{r} \times \dot{x} x^{n+1}$, and $\chi = \frac{cx^{n+2}}{r \times n + 2}$, Th. if $n = + \frac{1}{2}$ (as in the Common Parabola)

$\chi = (\frac{2cx^{\frac{1}{2}}x^2}{5r}) = \frac{1}{5}\pi d^2 b$; Th. is to the Cylinder as 4

to 5. And the Solid generated by the Infinite Asymptotic Space (in the Apollonian Hyperbola, where $y = x^{-1}$) about that Asymptote, is $= \frac{1}{2}\pi dxy$; Th. is to the Cylinder (the Rad. of whose Base is x, and Alt. = y) as 2 to 1. So that, in this Case, an Infinite Space generates a Solid of a Finite Dimension.

4. Where

4. Where the Axis of Rotation is parallel to the Abscissa, and the Plane adjacent to it, or to the Abscissa; (1) $\dot{\chi} = \dot{x}p - y \dot{y} \times \frac{c}{d}$, or (2) $\dot{\chi} = \dot{x}p^2 - p - \dot{A} \times \frac{c}{d}$.

Note, Where the Ordinates are Oblique to the Diameter, or Axis of Rotation, or where they respect a determined Point; they are to be reduced to other equivalent ones, parallel between themselves, and right to the Axis of Rotation.

Hence in Cask Gauging, let B, H , be the Bung, and Head Diameters, L , the Length, n the Number of Cubic Inches in the Gallon; and the Diam. : Periph. :: 1 : π .

Then the Contents of a Cask taken as the middle Frustrum of an Acute Parabolic Conoid, or of an Oblong Spheroid, is

$$L \times 2B^2 + H^2 - \frac{2}{3}B - H \div \frac{12n}{\pi}, \text{ or } L \times 2B^2 + H^2 \div \frac{12n}{\pi}$$

Th. in a Spheroidal Cask, when not Full, the Axis being Perpendicular to the Horizon, w = wet part of L , $d = \frac{1}{2}L \propto w$; if w be $>$, or $< \frac{1}{2}L$, the Cask is more, or less

than $\frac{1}{2}$ Full by $3B^2d - d^3 \times B^2 - H^2 \div \frac{1}{4}L^2 \div \frac{12n}{\pi}$, and therefore the Liquor contain'd therein must be

$$\frac{1}{2}L \times 2B^2 + H^2 \pm 3B^2d - d^3 \times B^2 - H^2 \div \frac{1}{4}L^2 \div \frac{12n}{\pi}$$

But in a Cask not Full, whose Axe lies parallel to the Horizon, and the Liquor cutting the Head; Let w = wet part of B , S = Segment of a Circle (whose Area = 1) to the versed Sine $w \div B$, and C = Content of the Cask, then will SC be the Quantity of Liquor remaining.

80. Of Projection.

The Impression or Representation of a Surface, on a Plane is called the *Projection* of that Surface: If the Impression be made by parallel Lines, 'tis called an *Orthographic Projection*; But if by Lines intersecting in the same Point, 'tis called a *Stereographic Projection*.

1. In

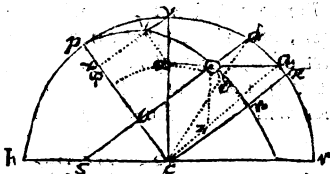
1. In the *Orthographic Projection*, 'tis evident that, 1. A right Line Perpendicular, Parallel, or Oblique to the Orthographic Plane is projected into a Point, an Equal, or a Shorter Line: And parallel Lines, tho' oblique to the Plane, are parallel when projected: 2. A Surface perpendicular to the Plane is projected into a right Line, equal to that intercepted between the projecting Lines: But a Surface parallel to the Plane, into an equal and like Surface. 3. Any *Conic Section* Oblique to the Plane is projected into another of the same Kind; Hence a *Circle* into an *Ellipse*. 4. The Representation of any Point of the Surface of the Sphere, *Orthographically* projected upon a Plane bisecting it, is distant from the Centre, by the *Sine* of the Arc from the Point to be projected to the Vertex of that Sphere.

Hence ca is Equivalent to ra , cb to zd . And all Circles Perpendicular, or Parallel to the Plane, are projected into Right Lines, or Equal Circles; but all other into *Ellipses*. Theres. all right Lines passing thro' the Centre of this Projection, are measured from the *Sines*; and all others are reduced to such; As for Instance,

Given the Latitude of the Place, Sun's Declination and Altitude; Reqd. the Sun's Azimuth, and Hour of the Day; the Sphere being *Orthographically* projected upon the Plane of the Meridian.

Let ra = Co-Latitude, ra or ca = $\odot$'s Altitude, zd or cb = $\odot$ Declination: Draw aa , db the Parallels of Alt. and Decl. draw $\odot z$, $\odot u \parallel ac$, $6c$; then cz , cu = Sines of the like Azimuth from the West, or Hour from 6, $\odot a$, $\odot b$.

To describe the Representation of Oblique Circles on this Projection; As suppose, to describe a Meridian, passing thro' the n th. Degree of the Equator, on the *Analemma*.



'Tis but making $cz : cn :: \chi v : \chi v :: 6d : 6\odot$, and the Points v , $\odot$, are in the *Ellipse* sought. Or the whole *Ellipse* (according to the Method in *Art. 68.*) is readily drawn by means of the *Foci*, (ϕ , t) found by cutting the Transverse Axe from n with pc its half.

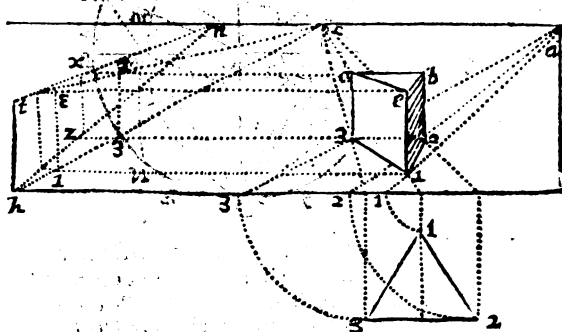
2. In

2. In the *Stereographic Projection*, that Point in which all Lines from the Extremes of the thing projected do concur is called the *Projecting Point*; thoſe Lines the *Projecting Lines*; the Plane cutting them, the *Stereographic Plane*, or *Plane of Projection*: And the Stereogr. Plane is ſuppoſed (if not otherwiſe mention'd) perpend. to another which is alſo ſuppoſed paral. to the Horizon, having thereon, the *ſear* or *Ichnography* of the thing to be projected, and called the *Original* or (by ſome) the *Geometrical Plane*. A Plane paſſing thro' the projecting Point, || to the Horiz. or perp. to it and to the Pl. of Proj. is call'd the *Horizontal*, or *Vertical Plane*. The com. Sect. of the Horiz. Plane, and Plane of Proj. is called the *Horizontal Line*, And the com. Sect. of the Vertical and Horiz. Pl. is called the *Axis of Projection*. The Interſect. of the Axis of Project. and the Horizontal Line is called the *Centre of Projection*. The com. Sect. of the Stereographic and Original Plane is called the *Base-Line*. The Interſect. of the Horiz. Line, and another drawn from the projecting Point parallel to a given right Line (L) is called the *Projecting Centre of the Line L*.

In this *Projection*, 'tis manifeſt, that, 1. The Projection of a Point, or Right Line is a Point, or Right Line. 2. The Projection of a right Line, on or above the Original Plane, paral. to the Base-Line is parallel to it in the Pl. of Projection: And the parts of its Projection, are :: 1 to thoſe of the Line it ſelf. 3. The Projection of a right Line $\perp$ or inclin'd to the Original Pl. and || to the Pl. of Proj. is $\perp$ or ſo much inclin'd to the Base Line: And the Projections of the equal parts of a Line perp. to the Pl. of Project. are unequal; and thoſe of the moſt remote are the ſmalleſt. 4. The Projections of parallel right Lines on or parallel to the Orig. Pl. and inclin'd to the Plane of Project. being produced, ſhall all paſs thro' the ſame Point in the Horiz. Line. 5. Theref. the Projection of right Lines perpend. to the Pl. of Project. do paſs thro' the Centre of Projection. 6. The Projections of equal Lines perp. or equally inclin'd the ſame way, on the ſame right Line (L) inclin'd to the Pl. of Projection, are limited by two Lines, which being prolong'd, ſhall paſs thro' the Projecting Centre of the Line L. 7. The Projections of Points above, or below the Horiz. Plane, are above, or below the Horizontal Line; and the more remote

Situation, with Respect to the Eye and to the Plane on which 'tis to be represented are, given.

As in the following Instance, where 123 is the Base of a *Triangular Prism*, whose Altitude is *bt*, to be put into Perspective; *c* the Centre of Projection or Point of Sight; *d* the Projecting Centre or Point of Distance: The Pra-



rice is evident from the foregoing Rules.

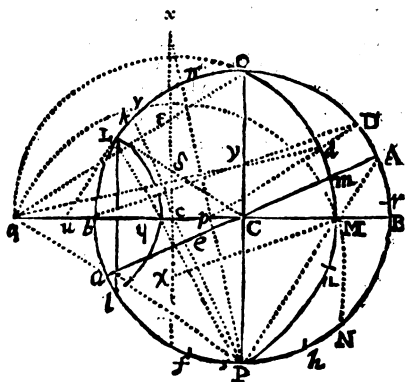
Note, That 'tis indifferent what Point of the Horizontal Line be taken, for determining the Elevations; it may as well be at *n* as at *c*; for $zx \parallel a_3$, bec. $nx : xt :: (ca : at :: c_3 : _3h ::) nz : zh$, th. $zx (\parallel ht) \parallel a_3$.

9. The *Stereographic Projection* of any Point (*A*) of the Surface of a *Sphere*, from the Pole of a Plane bisecting it, upon that Plane, is distant from the Centre by the $\frac{1}{2}$ Tangent of the Arc from the Point to be projected to the Pole opposite to the projecting Point; Thus, *M* is the Projection of *A*; th. *BM equiv. BA*, *MC equiv. Ao*, and *cA equiv. Md*.

1. Here all Circles, passing thro' the projecting Point, are projected into *Right Lines*; But all others into *Circles*: Since the Plane of the Projection cuts the Surface of the Cone form'd by the projecting Lines, either parallel, or (bec. $\angle BqP = \frac{1}{2} \text{arc } BP - \frac{1}{2} \text{arc } ba$ (by 54) $= \frac{1}{2} \text{arc } Pa = \angle aAP$) subcontrary to its Base. And the Angles in this Projection are equal to their correspondent ones on the Sphere.

Na

2. If



2. If Aa be the Diameter of a great Circle to be projected, whose Obliquity is AB (or ω); Let $o\pi = AB = \pi k$, draw $Pp\pi$, Pck , p shall be the Upper Pole (bec. $\pi A = Bo = 90^\circ$), and c the Centre of the Projected Oblique

Circle PMo ; since $cM (=t, \omega \div t, \frac{1}{2}90 - \frac{1}{2}\omega = s, \omega$ (by 65.16) $= cP$. And if $Pb = AB$ or $o\pi$, a right Line thro' P and b will project the Under Pole in the produced Plane.

Also the Centre of a Lesser Circle (Lyl) perpendicular to the Plane, is found by drawing CL , and $Lu \perp$ to it, then will u be the Centre reqd. Since $uy (=s, \omega - t, \frac{1}{2}90 - \frac{1}{2}\omega = t, \omega$ (by 65.16) $= Lu$.

The Radius and Centre of any Circle whose Distance from its Pole is d , may easily be found thus; From (P) the projected Pole, on a great Circle (whose Centre is c) passing thro' it, d being set off, finds a Point. (p), whose Tangent, terminated at π in a Line thro' c and P , or (which is the same) a Tangent to P , terminated at π in a Line thro' c and p , shall be the Radius, and πc the Distance from the Centre required.

3. The Centre χ of any great Circle (NMq) passing thro' M , shall be in cx , the produced Diameter of a Circle (PMo) bisecting the Primitive in P and o ; And (bec. Sph. $\angle PMN = \angle cM\chi$ by 56) $c\chi =$ Tangent of the Sph. $\angle PMN$ to the Rad. Mc .

4. Hence to draw the Circumf. of a great Circle thro' any two Points, (m, n), one, or both within the Primitive. Thro' (m)

(*m*) one of the Points in the Circle, draw a Diameter *Aa*, and $\pi b \perp$ to it, draw πm , let $bf = 2Ar$, a Ruler on π and *f* cuts *Aa* in *e*, thro' *e* draw $ev \parallel \pi b$, a right Line (*Bb*) bisecting the Distance between *m* and *n* at $\angle s$, will cut ev in the Centre sought. For the Centres of all great Circles passing thro' *m* are in the Line ev , and the Centre of a Circle passing thro' *m* and *n* is in the Line *Bb*, th. the Centre reqd. must be in both, *i. e.* in their Intersection *e*.

5. The Circumf. of a great Circle in this Projection, is Divided, or Measured, by a Ruler laid on either Pole, and the Divisions of the op. $\frac{1}{2}$ Periphery of the Primitive; For, since *B* equiv. *M* equiv. *C* equiv. *c* equiv. 90° . th. *BD* (equiv. *Cy* equiv. *cd*) equiv. *Md*. And the Circumf. of a small Circle (whose Dist. from its Pole is *d*) may be divided by Lines from that Pole to the Divisions in the Circumf. of a small Circle $\parallel$ to the Pl. of Proj. and descr. at the Dist. *d* from the under Pole of that Plane.

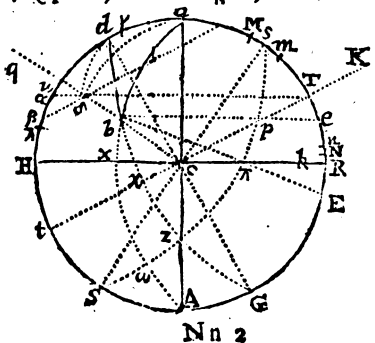
Or any Projected Circumf. is divided by Lines from the Divisions of its produced $\frac{1}{2}$ Periphery to either Pole.

6. A Spheric $\angle$ or the Inclination of the Planes of two great Circles is equal to the Distance of their Poles, or (which is the same) is measured by an Arc of a great Circle whose Pole is the Angular Point; and therof. is so in the Projection. And the Circumfs. of great Circles intersecting at $\angle s$, do mutually pass thro' each other's Poles.

7. The Δadb , and Δadd are the *Stereographic* and *Orthographic* Projections of the same Sph. Δ ; and their $\angle s$ *b* and *d* are in the same Radius *cu*. Let *scS* perp. *cu*.

Given *b*, Reqd. *d*; draw *Sbv*, and *vd* $\parallel$ *sS*.

Given *d*, Reqd. *b*; draw *Sdv* $\parallel$ *sS*, and *vbS*.



8. Let

8. Let $\nu\lambda = \nu s$, draw $S\lambda q$, with qs , from q , descr. a Circumf. $s\pi S$ cutting cT, cR, db, ab , in p, π, z, ω , then will p, π, c , be the Pole of db, ab, ad ; and b the Pole of $s\pi S$: Draw $bpe, b\pi E$, then $Ee = \pi p = sph. Lb$.

9. Draw $A\pi m, GpM$, make $mn = ma$, and $MN = Md$, a Line thro' A and n , G and N gives k, K , the Centre of the Arc abA, dbG .

10. Draw $\delta\alpha, \delta\beta \parallel HR, tT$, then will $Ha, t\beta equiv. bx, b\chi$.

11. And in the Sph. $\Delta c\pi p$, $Lc = ad$; $Lp = \chi z = db equiv. d\beta$; $L\pi = Supl. x\omega = Supl. ab = Ab equiv. Aa, i.e.$

$\left\{ Lc, Lp, L\pi \right\}$ in $\Delta c\pi p$, is $= \left\{ ad, db, Supl. ab \right\}$ in Δabd .
 $\left\{ p\pi, c\pi, cp \right\}$ in $\Delta c\pi p$, is $= \left\{ Lb, La, Supl. Ld \right\}$ in Δabd .

12. Hence (bec. $ad + d\beta + Aa = 2 Ls + a\beta$) the $3 Ls$ of a Sph. Δ is greater than $2 Ls$.

10. The Projection of any Point (P) of a Hemisphere, from the Centre (C) of the Sphere, upon a Plane touching it (in T) is distant from the Point of Contact (T) by the Tangent of the Arc PT; And this is called the *Gnomonic Projection*; where 'tis plain, that, 1. All great Circles on the Sphere, perpendicular to the Plane, are projected into *Right Lines*; all others, (as they are parallel to, touch, cut, or neither, a great Circle parallel to the Plane) into *Circles, Parabola's, Hyperbola's, or Ellipses*.

Hence, if the Sun moves in a Circle parallel to the Equator, the Shade of the Centre will describe a Conic Surface; And therefore, upon a Plane parallel to a great Circle, touching, or cutting the Conic Surface, or neither, will describe the Curve of a *Parabola, Hyperbola, or Ellipse*. Theref. on any Plane where the Sun sets, does not set, or only touches, the Projection of any Parallel of Declination is an *Hyperbola, Ellipse, or Parabola*.

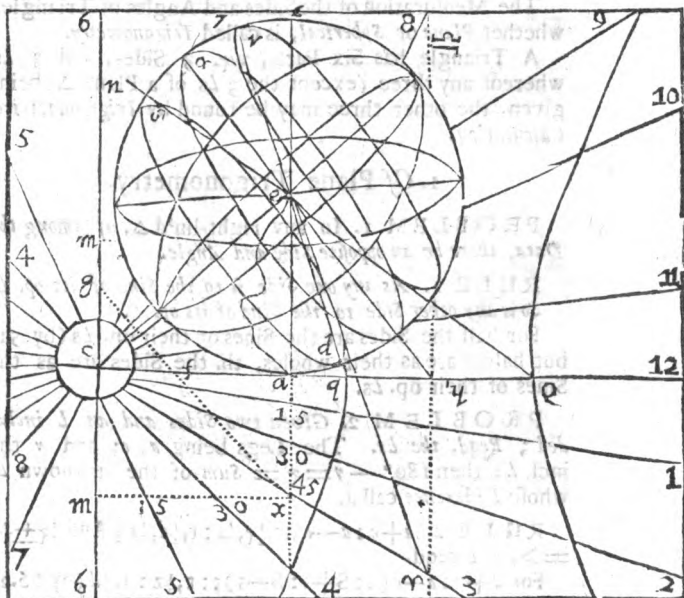
Thus, if ce, ca be the *Stile*, and *Substile*, *Dec* or *dec* the *Co-declinaz.* of the Sun; then,

If $e\delta$ produced be parallel to the Plane, and ed produced cut it in q , let $en = eq$, from n with nq cut qc , the Projected *Meridian* produced, in $Sup. \pi$, then $\pi q = Parameter$, and $\frac{1}{2}\pi q$, set from q the *Vertex*, gives the *Focus f* of the *Parabola*: If ed produced cut ca in q , and $e\delta$ produced cut it in Q , above, or below the Vertex e , let $en = eq$, bisect Qq in y , make $yf = y\phi = \frac{1}{2}nQ$, then shall f, ϕ be the *Foci* of the *Ellipses*, or opposite *Hyperbola's*.

Sola's deſcribed by the Shade of the Point *e*, their *Transverſe Axe* being *Qq*, and *Diſtance of the Foci* *nQ* (by 69.6)

Hence the *Parallels* of the Sun's Declination, &c. in the *Gnomonic Projection*, are eaſily deſcribed (by 68') upon an *Horizontal Plane*, in any Latitude.

And all *Planes* are parallel to ſome great *Circles*, which alſo are *Horizons* in ſome place or other; therefore all *Planes* on which *Dyals* are made, are *Horizontal Dyals* on ſome part of the Earth.



The Method of projecting the *Hour Circles* Gnomonically, on an *Horizontal Plane*, &c. or the Practice of *Dyalling* is hence alſo evident; Thus, let *c12* be the Projection of the *Meridian*, and (*ob. 1* to it,) that of the 6th *Hour Circle*; make *Lace* = *Stile's Height*, and *af* = *Sine* thereof, to any *Rad. ca*; let *ax* = *cm* = *af*, on *ax*, *mx*, from *a*, *m*, ſet the *Tangents* of the *Hour L* at the *Pole*, from *12* to *3*, and *6* to *3*, to the *Rad. ax*, *mx*; Lines drawn

drawn from c thro' those Points, shall be the *Hour-Lines* reqd.

For $\begin{cases} ca : r :: ax : t, acx. \text{ and } cm : r :: mx : t, mcx \\ fa : r :: ax : t, afx. \text{ and } gm : r :: mx : t, mgx \\ th. ca : fa :: t, afx : t, acx. \text{ and } cm : gm :: t, mgx : t, mcx \\ i.e. Rad. : S, \text{ Stile's Height} :: t, \text{ Hour } L \text{ at the Pole} : t, \\ \text{Hour } L \text{ on the Plane.} \end{cases}$

81. Of Trigonometry.

The Mensuration of the Sides and Angles of Triangles, whether *Plane* or *Spherical*, is called *Trigonometry*.

A Triangle has Six Parts, viz. 3 Sides, and 3 $\angle$ s, whereof any three (except the 3 $\angle$ s of a Plane Δ) being given, the other three may be found by *Trigonometrical Calculation*.

1. Of Plane Trigonometry.

PROBLEM 1. In any right-lin'd Δ , if among the Data, there be an opposite Side and Angle.

RULE 1. As any one Side is to the Sine of its op. $\angle$. So is any other Side to the Sine of its op. $\angle$.

For half the Sides are the Sines of their op. $\angle$ s (by 51) but halves are as their wholes, th. the Sides are as the Sines of their op. $\angle$ s.

PROBLEM 2. Given two Sides and an $\angle$ included; Reqd. the $\angle$ s. The Legs being a, c ; and y the incl. $\angle$; then $180^\circ - y = \angle = \text{Sum of the unknown } \angle$ s whose Difference call d .

RULE 2. $a + c : a - c :: t, \frac{1}{2}\angle, Ls : t, \frac{1}{2}d, Ls$; And $\frac{1}{2}\angle \pm \frac{1}{2}d = >, < \angle$ reqd.

For $a + c : a - c (:: S + s : S - s) :: t, \frac{1}{2}\angle : t, \frac{1}{2}d$ (by 66.2)

Note, If $\angle y = \angle$, then $a : r :: c : t, \text{op. } \angle$, or $c : r :: a : t, \text{op. } \angle$.

For the *Tabular Parts* of any two Lines are proportional to their Parts according to any other Measure.

PROBLEM 3. Given the three Sides; Reqd. the $\angle$ s. Let a, c be the Legs of the $\angle$ reqd. and $a + c = \angle$, or d , call the Base or Side op. b , and the Diff. of the Segments of b , made by a perp. from the $\angle$ sought, call δ .

RULE

$$\delta^2 + 2n\delta = \overline{p^2 + n^2(a^2)} + d^2 + 2ad, \text{ Th.}$$
$$2n+d : 2a+d :: d : d) \text{ And } \frac{1}{2}b + \frac{1}{2}d =$$

$\triangleright, \triangleleft$ Segment; whence the $\angle$ s may be found

Or bec. $r : s :: c : x :: 2a \times c : (2a \times x) \pm a^2 \pm c^2 \mp b^2$

(by 64. 17) th. $2ac : (2ac \pm a^2 \pm c^2 \mp b^2) r^2 - b^2$, or $b^2 -$

$$d^2 :: r : (r \pm s) v, \text{ or } v :: \frac{1}{2} r^2 : (\frac{1}{2} r v) s^2, \frac{1}{2} L, \text{ or } (\frac{1}{2} r v)$$
$$s^2, \frac{1}{2}L \text{ (by 65.14 \& 12) th. } 4ac : r^2 :: z^2 - b^2 : s^2, \frac{1}{2}L ::$$
$$b^2 - d^2 : s^2, \frac{1}{2}L. \quad \text{And } z^2 - b^2 : b^2 - d^2 :: s^3, \frac{1}{2}L : s^2, \frac{1}{2}L$$
$$\therefore r^2 : t^2, \frac{1}{2}L \text{ (by 65. 10)} :: t^2, \frac{1}{2}L : r^2 \text{ (by 65. 5) Let } b =$$

$\frac{3}{2}a + b + c (\frac{1}{2}a + b)$ then will these Rules follow, viz.

1. $(4ac : \overline{a+b} \times \overline{a-b} ::) ac : b \times \overline{b-b} :: r^2 : s^2, \frac{1}{2}L$

$$2. (4ac : b+d \times b-d ::) ac : b-a \times b-c :: r^2 : s^2, \frac{1}{2}L$$

3. $(\overline{a+b \times a-b} : \overline{b+d \times b-d} ::) b \times b - b : b - a \times b - c :: r^2 : t^2, \frac{1}{2} L$

4. $(\overline{b+d \times b-d} : \overline{z+b \times z-b} :: \overline{b-a \times b-c} : \overline{b \times b-b} :: r^2 : r^2)_{\frac{9}{2}} L$

And if $L = \text{Logarithm Sine}$, and $l = \text{Arith. Compl. of any Log. Sine}$, then, for Practice,

$$l_a + l_c + L, \frac{1}{2}b + d + L, \frac{1}{2}b - d \div 2 = L, \frac{1}{2}L \text{ reqd.}$$

To the various Uses of *Plane Trigonometry*, may be added, that,

If an Equation be $x^2 \pm ax = b^2$;

Make b , and $\frac{1}{2}a$ the Legs (AB, BC)

of a $\triangle$, Then Hypotenuse $= \frac{1}{2}a$

(i : e EB, or BD) = x. Or bec.

$b:r::\frac{1}{2}a:t, \frac{1}{2}L$, Th. $r:b::t,L:x$, or $s,L:b::s,L:x$.

If the Equation be $-x^2 - ax = b^2$; Make b , and $\frac{1}{2}a$ the Base and Height (AB, AC) of a Δ , then Erect

the *Perp.* and *Hypot.* (AB, AC) of a $\triangle$, then *Hypotenuse* $\pm$ *Base* (i.e. BD BE) = *Greatest* or *Least Root*

nu/c $\pm$ ba/je (i. e. BD, BE) $\equiv$ Greatest, or Least Root.
Th. Bec. $1:1::b:s$ // th. s or $s':b:s$ or $s':s$

In. Dec. $\frac{1}{2} 1: r :: a: s, \frac{1}{2} 4$, th. s, of s, $\frac{1}{2} 4: b :: s$, of s, $\frac{1}{2} 4: a$
the $<$ or $>$ Root.

the $< 0 \mid >_{\text{root}}$.

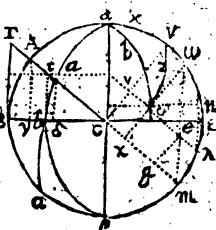
2. Of Spheric Trigonometry.

In Δ CAB, Cab, Orthographically projected,
 Sup. H, P, & B, the Hypotenuse, Perpendicular, and Base.

1. $\left\{ \begin{array}{l} s, H : s, P :: s, b : s, p :: R : r, \\ s, B : t, P :: s, b : t, p :: R : t, \end{array} \right\} \angle \text{ at the Base.}$
 For $CA : Ay :: ca : ad$; And $CB : TB :: cb : tb$.

2. In any Sph. Δ ($b\alpha\beta$), The Sines of the Sides are as the Sines of their op. Ls. For (by prec.) $R : s, \beta :: s, \beta b : s, Bb$, and $R : s, \alpha :: s, \alpha b :: s, Bb$; Th. $s, \beta : s, \alpha :: s, \alpha b : s, \beta b$.

3. Theref. in Right $\angle$ d Sph. Δ s, of the three Parts which (besides the $\angle$) enter the Question, let that be called the *Middle* to which both *Extremes* are either *Conjunct* or *Disjunct*; Then,
Rad. + Log. of the Middle = Logs. of the Extremes.



If on	Middle.	Extr. Conj.	Extr. Disj.
Hyp. or $\angle$	Co-Sine.	Co-Tangent.	Sine.
Leg.	Sine.	Tangent.	Co-Sine.

Hence all the Cases are easily solved.

For the *Middle* part may be one of the *Legs* (b, p); the *Compl.* of the *Hypotenuse* (b); or the *Compl.* of either of the *Oblique Ls* (e, e); In each, there are three Cases: As sup. the *Middle*, be b, Le, b .

1. ($r : t, Le ::$) $s, b : t, p$ (by 1) $Le : r$ (by 65.5)

2. ($r : t, \beta (t, b) :: s, \epsilon$) $s, e : (t, n) t, b :: t, b : r$ (by 1.)

3. ($r : t, Le :: s, n$) $s, b : (t, \epsilon) t, e$ (by 1) $t, e : r$ (by 65.5)
 Th. $R \times S, \text{Middle} = \square \text{ Tangents of the Conj. Extremes.}$

1. $r : (s, b) s, n :: (s, Le) s, \chi : s, b$ (by 1)

2. $r : (s, Le) s, \chi :: (s, \tau) s, p : (s, \epsilon) s, e$

3. $r : (s, \tau) s, p :: (s, \beta) s, b : s, n (s, b)$

Th. $R \times S, \text{Middle} = \square \text{ Co-Sines of the Disj. Extremes.}$

Note, In $\angle$ Sph. Δ , if the *Legs* (and th. the *Ls*) be like, or unlike, the *Hypotenuse* is $<$, or $> 90^\circ$. and the contrary.

If the *Hypotenuse* is $<$ or $> 90^\circ$, either *Side* will be like, or unlike its adjacent *L*; and the contrary.

In *Oblique $\angle$ d Spherical Triangles*; Suppose the *Obl. Δ* to be divided into $\angle$ d Δ s, by a *perp.* let-fall from one End of a given Side, (whose other End is adjacent to a given

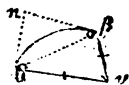


given L) within, or without the Δ , as the L s at the Base are of the same, or of different Kind. Then, if there are two parts given in one Δ , corresponding to the two given and sought in the other, the perp. being a 3d. in each, will denote a Middle or Extreme Part; And (rejecting the Perp.) the Logarithms of the op. parts are equal. But if 4 such Parts don't correspond (as it happens when the perp. falls from or on a part given or required;) find (by 3) a part (in the $L\Delta$ wherein a Side and an L are given) of the given or sought one, from or on which the perp. falls, Then you'll have the 4 parts reqd. for the Second Operation.

When two Sides, or L s, and the L , or Side included, are given; the other L s, or Sides may be found, without setting fall a perp. by these Proportions.

$$1. s, \frac{1}{2}r, cr : s, \frac{1}{2}d, cr :: ct, \frac{1}{2}L : t, \frac{1}{2}d, Ls.$$

$$2. cs, \frac{1}{2}r, cr : cs, \frac{1}{2}d, cr :: ct, \frac{1}{2}L : t, \frac{1}{2}r, Ls.$$



For, let $vb\beta$ be the Obl. Spb. Δ Stereographically projected, if the Right Line $b\beta$, and the Tangent sbu , gn , be drawn, it appears that the Spb. $L vb\beta - vb\beta = \text{Plane } L vb\beta - vb\beta$, Th. ($r, cr : d, cr ::$) $cr, \frac{1}{2}Lv : t, \frac{1}{2}d, Ls$ (by 81, 1. Fr. 2.) $:: (r, \frac{1}{2}t, cr : d, \frac{1}{2}t, cr ::) s, \frac{1}{2}r, cr : s, \frac{1}{2}d, cr$ (by 66.) And bec. $d, s, cr : r, s, cr :: d, s, Ls : r, s, Ls$ (by 81, 2, 1.) th. (by 66) $t, \frac{1}{2}d, cr (s, \frac{1}{2}d, cr \div cs) : t, \frac{1}{2}r, cr (s, \frac{1}{2}r, cr \div cs) :: t, \frac{1}{2}d, Ls : t, \frac{1}{2}r, Ls$, Th. $cs, \frac{1}{2}r, cr : cs, \frac{1}{2}d, cr :: ct, \frac{1}{2}L : t, \frac{1}{2}r, Ls$. This Demonstration, with several other New and Valuable things of this Nature, I had from the Excellent Geometer Mr. Halley, whose Freedom in Communicating, and Readiness in Assisting, I shall always own with the highest Gratitude.

4. The two Cases excepted, viz.

1. Three Sides being given; the Angles reqd. Let x, r be the Legs of the L sought, and b the Bif., or Side op.

Then $1, x + \frac{1}{2}r + L, \frac{1}{2}b + x + \frac{1}{2}r + L, \frac{1}{2}b \text{ or } x + \frac{1}{2}r \div 2 = \text{Log. } cs$, or $s, \frac{1}{2}L$. For in Δnol , (Fig. 1. p. 280c.) $Lo = me = x$;

$s, Ln = s, \frac{1}{2}b + x + \frac{1}{2}r$; $n\lambda = b \text{ or } x + \frac{1}{2}r$; ol equiv. $\chi^m = \text{Arc } om = \text{Suppl. } Lv$; $me (\frac{1}{2}m\omega = s, \frac{1}{2}\text{Arc } m\omega) = cs, \frac{1}{2}Lv$; Draw $eg \parallel \chi\omega$; $v\lambda (s, v\lambda) = s, r$; Then $s, o : s, n :: \frac{1}{2}n\lambda : \frac{1}{2}ol$, and $v\lambda : \frac{1}{2}ol :: r : mg) :: r^2 : (r \times mg) me^2$. Th. $s, o \times v\lambda : \frac{1}{2}n\lambda \times s, n :: r^2 : me^2$, that is,

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$s, x \times s, r$

$s, x \times s, r :: s, \frac{1}{2}b \cos x + r \times s, \frac{1}{2}b + x + r :: r^2 : cs^2, \frac{1}{2}Lv$.
And after the like manner 'tis prov'd that

$s, x \times s, r :: s, \frac{1}{2}b \cos x \times s, \frac{1}{2}b + x \times r :: r^2 : s^2, \frac{1}{2}Lv$.

2. *Three Angles being given; the Sides required.* Instead of the greatest L , take its Supplement, and call the L s Sides, and the Sides L s; then do as in the preceding Case.

82. The Principles of Mechanics.

1. *Velocity (v) or that Affection of Motion, whereby a Body runs a given Space (s) in a given Time (t) is the Ratio of the Space to the Time, i. e. $v = s \div t$; Th. $vt = s$, and $VTs = vtS$. Th. if $V : v :: T : t$, then $S \times t^2 = s \times T^2$, or $S \times v^2 = s \times V^2$.*

2. *Moment (m) or that which conduces to the effecting of Motion, is compounded of the Velocity (v) and Quantity of Matter or Weight (w) i. e. $m = vw$; Th. $mVW = Mvw$. And bec. $V = S \div T = M \div W$, th. $TM : tm :: WS : ws$.*

That Motion is said to be *Equable*, which runs over all the Parts of Space with the same Velocity; but *Accelerated*, or *Retarded* when its Velocity is continually *Augmented*, or *Diminish'd*. The *Innate* or *Natural Force* of a Body is that by which it endeavours to persevere in its State of Rest, or uniform direct Motion. An *Impress'd Force* is an Action exercis'd on a Body to change its State of Rest, or Motion. *Centripetal Force* is that by which a Body is Impell'd, or Attracted towards some Point as a Centre. *Centrifugal Force* is that by which a Body endeavours to recede from its Centre.

3. *All Bodies will continue in their State of Rest, or uniform direct Motion, unless they are compell'd to alter that State by some Force impress'd upon them.*

4. *The Change of Motion is ever proportional to, and its Direction is in the same right Line with the Impress'd Force.*

5. *The Actions of two Bodies upon one another are always equal, and have contrary Directions.*

6. *Hence, if a Body g (in Fig. 6.) be impell'd by two different Forces (F, f) to move (in $gd, g\lambda$) with an uniform Velocity, it will describe the Diagonal (gh) of a Parallelogr.*

in

in the same time as it would describe the Sides ($g\delta, g\lambda$) by F, f , separately, For, because the Force F, f (in $g\delta, g\lambda$) hinders not the Velocity of the Force f, F from carrying the Body (g) to a Line ($\lambda h, \delta h$) paral. to the Direction of F, f , in the same time; whether the Force F, f be impress'd or not; th. (g) will be found in each of the paral. i. e. in their Interfection.

7. Whence we have the Method of Compounding and Resolving any given Directions. For any Motion may be consider'd as Compounded of others, and th. may be Resolved into them.

8. If a Body be held immoveable by two equal Forces acting with contrary Directions, since either of these Forces may be resolv'd into two others, therof. 'tis the same as if the Body was held by three different Forces; And these three Forces are one to the other, as Lines drawn parallel to the respective Directions, and terminated at their mutual Concourse.

9. Also the Proportion of an Oblique Force to move a Body, is to that of the same Force coming with a Perpendicular Direction, as the Sine of the L of Incidence, is to the Radius. For the Oblique Force is compounded of two Forces, the one $\parallel$, and the other $\perp$ to the Surface of the Obstacle, which is only affected by the latter or Sine of Incidence, the Oblique Force being made Radius.

10. Hence the Forces of a Fluid Medium on a Plane cutting the Direction of its Motion at different Inclinations, are as the Squares of the Sines of the L s of Incidence. For the Force of a Particle is as the Sine of Incidence; And the Number of Particles, that strike in equal time, is as the Sines of Incidence; Th. the Forces of all the Particles are as the Squares of the Sines of the L s of Incidence.

11. If the Velocity of a Medium be different, the Forces on a Plane cutting that Medium with the same Inclination, are as the Squares of the Velocity. For the Force of each Particle is as its Velocity; And the Number of Particles, that strike in equal time, is as their Velocities: Th. the Forces are as the Squares of the Velocity.

12. The Force of the Water upon the Rudder of a Ship in Motion is as the Sq. of the Sine of the Inclination of the Rudder to the Keel (by 11); And the Force of the Rudder upon the Keel is as the Co-Sine into the Sq.

of the Sine of that Inclination, i. e. as $r^2 - s^2 \Big|^{1/2} \times s^2$;

which if a Maximum, then $-s^3 \div r^2 - s^2 \Big|^{1/2} + 2ss \times r^2 - s^2 \Big|^{1/2} = 0$, Th. $s = \sqrt{\frac{2}{3}}r^2 (= 54^\circ, 44', 08'' \text{ near}) = \text{Sine of that Angle which the Rudder, in the most advantageous Position, should make with the Keel.}$

13. And the Force (f) of a Fluid Medium upon a $\triangle$ moving according to the Direction of its Base (b) is to the Force (F) on the Circumscr. Parallelogr. as the Sq. of the Perp. (p) to the Sq. of the Hypotenuse (h) i. e. $f, \Delta : F, \square :: p^2 : b^2$. And the Force, striking with the Velocity (v) upon the Surf. descr. by p, taken infinitely small, about an Axe, at the Distance y, is as vp^3 (for py is as that Surf.) Th. $b^2 : p^2 :: vp^3 : vp^3 y \div b^2$ the Force on the Surf. descr. by b, at the same time.

Th. if H, b be any two adjoining Particles of a Curve,

$yp^3 \div b^2 + rP^3 \div H^2$ or $yp^3 \div p^2 + b^2 + rP^3 \div p^2 + B^2$ is the Force of the Fluid on the Surface generated by the Rotation of $H + b$; which if a Minimum, then (B, b

being variable) $2bby^3 \div b^4 = -2BB rP^3 \div H^4$ or (bec.

$B + b$ is constant, th. $\dot{b} = -\dot{B}$) $B rP^3 \div H^4 = by^3 \div$

b^4 , or $xy^3 \div \dot{x}^4$ or (sup. $P = p$) $xy \div \dot{x}^4 = \text{an Invariable: The Property of the Curve that generates the Surface of a Solid, which moving in a Fluid Medium, according to the Direction of its Axis of Rotation, shall meet with less Resistance than any other Solid generated by a Curve described to that Axe, and passing thro' the Extremity of the given Ordinate y.}$

And drawing $\omega \perp$ to the Axe, and $\tau \parallel$ to the Tangent, the $\triangle$ made by ω, τ, β is Sim. to $\triangle$ made by

p, b, b , th. $4\omega^2\beta(4\omega^2 \times \omega \dot{x} \div \dot{y}) : \tau^3 (\omega^3 \dot{x}^3 \div \dot{y}^3) :: \tau (\omega \dot{x} \div \dot{y}) : y$.

14. Let the unequal Radii CL, Cl (Fig. in p. 289.) sustain the Weights P, W, by the Cords LP, lW; Required their Forces to move the Wheel: Sup. (ω, w) $CD, Cd \perp$ to the Directions of P, W, with CD descr. a Circumf. cutting the Direction of W in g; Then (since a Body hanging freely by any Point, makes that Point as heavy as if

if it existed there) the Weight W is as gb , and its Force to turn the Wheel is as gd , And if $W : P :: (gb : gd :: gC \text{ or } CD : Cd, \text{ i. e. if } P \times \pi = W \times \omega, \text{ or when } P \times sp. P = W \times sp. W$, Then contrary Forces and Resistances will sustain one another; Th. if P be greater than

$\omega \div \pi \times W$ or $sp. W \div sp. P \times W$, the Power will overcome the Resistance or Weight, with a Force equal to the Excess. Hence any given Weight or Resistance, may be moved or overpower'd by any given Force: Consequently, the greatest conceivable Weight may be moved by the least conceivable Power. And,

15. In the *Leaver*, or *Steel-yard*; when $P \times \pi = W \times \omega$, the Power will sustain the Weight or the Weights will Equiponderate. Theref. in the *Ballance*, where $\pi = \omega$, the Weights must be equal.

This Property of the *Leaver* explains the Powers of *Oars* in *Rowing*; of *Iron Crows* in moving or lifting Weights; of the *Hammer* in drawing out Nails, &c. also of *Pincers*, *Shears*, &c. which consist of double *Leavers* bearing on a Common *Fulcrum*.

If two *Fulciments* (x, z) sustain a *Leaver* Horizontally, and c be the Centre of grav. of the Burden; Then (with respect to the fix'd Points x, z) $p, x : W :: zc : xz$, & $p, z : W :: xc : xz$, Th. $p, x : p, z :: zc : xc$; And x , or z will sustain $zc \div zx$, or $xc \div zx$ of W . Hence a Weight carried between two Men may be placed on the *Leaver* in any given Proportion; And so two Horses drawing any Weight, the Resistance may be divided according to the Strength of the Horses.

If two *Fulciments* (x, z) sustain a *Leaver* oblique to the Horizon, and a, c its Intersections with the 1^{st} to the Horizon, passing thro' the C. of Gravity of the Weight (W) sustain'd *Below*, or *Above* that *Leaver*; Then x, z , will bear

$az \div xz$, $ax \div xz$ part, or $cz \div xz$, $cx \div xz$ part of W . And if the Centre of Grav. be *Below*, or *Above* the *Leaver*, the *Superior*, or *Inferior* Fulciment will bear so much the greater part of W , as the *Leaver* is more inclin'd.

If a Weight (W) be placed on Three Beams (α, β, γ) joyn'd in p , and supported by Three Props α, β, γ ; suppose each Beam a Line, and produced thro' p , to meet the Sides of a Δ (whose angular Points are at α, β, γ) in a, b, c ; Then α, β, γ , will bear $pa \div aa$, $pb \div \beta b$, $pc \div \gamma c$ part of W .
Hence,

Hence, in *Building*, where the *Joysts* or *Girders* are too short, they may be placed so as to sustain each other; and by the Combination of such, any *Flooring*, or the like, may be compleated, and the Weight sustain'd at each Contignation may readily be computed.

Hence also, the Strength of Beams fix'd at one End is easily found: And that a *Parabolic*, or a *Prismatic Parabolic* Beam, has the same Resistance in all its parts; Theref. $\frac{2}{3}$, or $\frac{1}{3}$ of the Timber, will in this Case, do the same Service: Which in *Civil* and *Naval Architecture* is of no small Importance.

16. In the *Wheel*; when $\text{Power} \times \text{Rad. of the Wheel} = \text{Weight} \times \text{Rad. of the Windlace}$, the *Power* will sustain the *Weight*. Whence the Force of the *Capstan*, the *Crane*, &c. as also of all *Engines* with *Tooth'd Wheels*, may be accounted for.

17. In the *Pulley*; (let $n =$ Number of Falls or Cords at the Block hook'd to the Weight) when $P : W :: (Sp. W : Sp. P ::) 1 : n$, or when $P = \frac{1}{n} W$, the *Power* will sustain the *Weight*.

18. In the *Wedge*; the Weight w lying on the two Oblique Planes $at, \alpha t$ ($\perp Lt$) in Fig. p. 289. may be consider'd as a *Wedge*, where f, w on $\alpha t : f, \text{Hammer}$ (according to the Line wx) on that Plane $:: tx : wx$, and f, w on $at : f, w$ on $at :: tx : wt$. And when $\text{Force} \times \text{Altitude of the Wedge} = \text{Resistance} \times \text{Thickness}$, the *Power* will be Equivalent to the *Resistance*. This accounts for the Force of *Hatchets*, *Chisels*, and all *Edg'd* or *Pointed Instruments*, as also *Saws*, *Files*, &c.

19. In the *Screw*, which may be taken as a *Wedge* impell'd by a *Lever*; When the *Power* that turns the *Screw* $\times \text{Circumf. descr. by it} = \text{Resistance of the Obstacle}$ (to be press'd or rais'd) $\times \text{Distance between the Thrds of the Screw}$, then the *Power* will be Equivalent to the *Resistance*.

20. Let a Weight ($w = P$) be sustain'd partly by a *Power* at L , and partly by the *Inclin'd Plane* ab ; draw wx , and $wt \perp \text{Horizon}$, and *Plane*; then the descending Force of w may be express'd by wx , which may be resolv'd into the Forces wt, tx : Th. f, w descending along the Incl. Pl. $ab : f, w$ along the Perp. Pl. $as :: (tx : wx :: s, Lxwt : s, Ltwy ::) s, L$ of *Inclination*: cs, L of *Traction*.

21. Th. if a *Power* at L sustain a *Weight* (w) upon an *Inclin'd Plane*, by a *Direction* (Lw) which, passing thro' the

the C. of Grav. of w , meets the Plane; Then *Power* : *Weight* :: s, L Inclination : cs, L Traction. If $Lw \parallel ab$, Then (bec. Δwix Sim. Δbsa Sim. ΔsFa , if $sF \perp ab$) *Power* : *Weight* :: ($ix : xw ::$) $as : ab :: s, Inclinat. : Radius$. Th. at 30 Deg. Incl. the *Power* must be double of that which is necessary to draw a *Weight* upon an Horizontal Plane.

If $Lw \parallel sb$, then *Power* : *Weight* ($:: ix : wx$) :: $as : sb :: s, Incl. : cs, Incl.$ And *Power* at L , when $Lw \parallel ab$: *Power* at L , when $Lw \parallel sb :: sb : ab$.

22. If as, ab , be two Planes, the one $\perp$, the other Oblique to the Horizon, let $sF \perp ab$; Then the Initial Forces, or Celerities of Descents, acquir'd in the same time, are :: ($sa : aF ::$) $ab : sa$, i. e. are reciprocally as the Lengths of the Planes.

23. And in equal Times, *Sp. run in* $ab : Sp. run in as ::$ (*vel. in* $ab : vel. in as :: as : ab$) :: $aF : as$, i. e. The *Weights* shall descend thro' aF , and as in equal Times. And (drawing $\mu n \parallel sF$) the Spaces $\mu s, nF$ shall be pass'd over in the same Time.

Th. if the Diameter of a Circle be $\perp$ to the Horizon, a Body shall descend in the same Time thro' any Chord whatsoever conterminous to that Diameter.

24. Times in ab, as , are :: (Times in $ab, aF :: \sqrt{ab} : \sqrt{aF} ::$) $ab : as$, i. e. Times of Descents thro' Planes equally high are directly as the Lengths of the Planes. Th. (if $\mu F \parallel sb$) Times in $Fb, \mu s :: Fb, \mu s$.

25. And since v , in $s : v$, in $F :: as : aF :: \sqrt{ab} : \sqrt{aF} :: v$, in $b : v$, in F , Th. v , in $s = v$, in b , i. e. The Velocities acquir'd in falling thro' Planes equally high are equal.

26. Or, let a Body descend by a Line (l) inclin'd to the Horizon, with a Velocity increaſing as the Time (t .) Sup. b, c the Sine and Co-Sine of the Inclination, v the last Velocity;

Then (l and c being variable) $\dot{l} = cc \div l$, but v is as $\sqrt{b}$, th. t , in $\dot{l}$ is as $cc \div \sqrt{b} \times l$ or $cc \div \sqrt{b} \times \sqrt{b^2 + c^2}$, th. t , in $\dot{l}$ is as $\sqrt{b^2 + c^2} \div \sqrt{b}$ or $l \div \sqrt{b}$, th. t^2 is $l^2 \div b$, and $HT^2 l^2 = bt^2 L^2$, also (bec. $vt = l$) $HTlv = btlv$. Hence,

1. $T : t :: L \times \sqrt{b} : l \times \sqrt{H}$; And $V : v :: lHT : lbt$.

2. If $L : l :: H : b$, Then $T : t :: \sqrt{L} : \sqrt{l} :: V : v$.

3. If $H = b$, Then $T : t :: L : l$; And $V = v$.

4. If

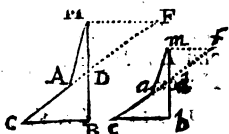
4. If $L = l$, Then $T :: \sqrt{b} : \sqrt{H}$, And $V :: v :: t : T$.
 5. If $L^2 b = l^2 H$, Then $T = t$, And $V :: v :: L : l$. Th.
 the Times of Descents thro' the Chords (L, l) of Arcs
 (whose vers'd Sines are H, h) are equal. For, in a Circle
 $L^2 : l^2 :: H : h$. (by 65.3.)

27. If a Body descends thro' how many soever contiguous
 Planes, however inclin'd, it shall acquire the same
 Velocity in the lowest Point, as if it had descended by
 the Perpendicular. Th. a Body descending by the Circumf.
 of a Circle, or any Curve Line, shall acquire in the
 Lowest Point, that Velocity which it would get by a Perp.
 fall from the same Height. And, bec. a Body thrown
 upwards with that Velocity which it got last by a Perp.
 fall, does ascend to the same Height, a Body if carried
 upwards (with the Velocity acquir'd in the Lowest Point,
 in descending by any Curved Surface) by the same, or any
 other Surface, however inclin'd, shall ascend to the same
 Height from whence it came, and have, in Points equally
 high, the same Velocity.

28. The Velocities of a Pendulous Body (at s) describing
 different Arcs (sf, sF) are in the lowest Point (s) as the Chords
 of those Arcs. For (drawing $cf, \mu F \perp as$), v , in s are as ($\sqrt{cs}$
 $: \sqrt{\mu s} :: \sqrt{csa} : \sqrt{\mu sa} :: sf : sF$).

29. And (since the Direction of the Impetus of a Body
 descending by a Curve Line is Horizontal in the
 lowest Point) if a Pendulous Body be struck in the lowest
 Point (s) by an Horizontal Force equal to that in falling thro'
 μs , it shall ascend from thence to the same Height. Since
 the smallest Arcs coincide with their Chords, which in
 the lowest Point are Isosceles, Th. the smallest unequal
 Oscillations are perform'd in the same Time. But greater
 Arcs are not as their Chords, Th. Longer Pendulums, that
 describe fewer Degrees, err less; And greater Arcs take up
 a little more time than shorter ones do.

30. Let MAC, mac be several contiguous Planes, equally
 inclin'd to one another, and to the Horizon, and of Proportional
 Height; Then the Times in
 which a Body runs thro' those Planes,
 shall be in a Subduplicate Ratio of
 their Altitudes MB, mb . For (t, MA
 $: t, ma :: \sqrt{MA} : \sqrt{ma} :: \sqrt{FA} : \sqrt{fa}$
 $:: \sqrt{FC} - \sqrt{FA} : \sqrt{fc} - \sqrt{fa} ::$
 $t, AC : t, ac :: t, MA + t, AC, &c. :: t, ma + t, ac, &c. ::$
 $\sqrt{MB} : \sqrt{mb}$. Th.

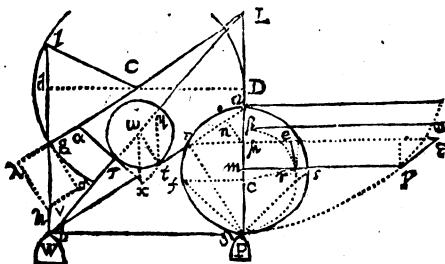


Th. in two Pendulums describing like Arcs (if $L, l =$ Lengths, $T, t =$ Times, $N, n =$ Number of Vibrations in the same Time) $T : t :: \sqrt{L} : \sqrt{l}$; And (since $NT = nt$) $N : n (:: t : T) :: \sqrt{l} : \sqrt{L}$, or $N^2 : n^2 :: l : L$. Th. if $L = 39,125$ Inch. and $N = 60$ Vibrations in 1'' of Time; Req'd. l , so as to make n or sup. 100 Vibrations in that Time;

Then $l = LN^2 \div n^2 = 39,125 \times 60^2 \div 100^2$. Also if $l = 20$ Inches, and n the Vibrations in 1'' be requir'd, Then

$$n = \sqrt{N^2 L \div l} = 60^2 \times 39,125 \div 20 \Big| \frac{1}{2}.$$

31. *The Times, in which a Weight descends from any Point (π) of a Cycloid, are equal among themselves: And have to the Time of a Perpendicular fall thro' (as) the Axis of the Cycloid, the Ratio of $\frac{1}{2}$ the Circumf. of a Circle to the Diameter.*



For, let $\pi\beta \perp sa$, and any Rt. Line $pm \parallel \pi\beta$, cutting the $\frac{1}{2}$ Circumf. on $sa, s\beta$ in q, r ; let pe and re touch the Cycloid and Circle in p and r , draw $em \parallel pm$, and suppose 'em infinitely near, Then will re , and pe coincide with their respective Arcs; But $pe : m\mu (:: es : sm :: a (sa) : s\beta$ (by 65.2.) $:: \sqrt{a} : \sqrt{sm}$, th. $pe = m\mu \times \sqrt{a} \div \sqrt{sm}$, and vel. in

p or m is as $\sqrt{m\beta}$, th. t , in pe is $(\frac{pe}{\sqrt{m\beta}} = \frac{m\mu \times \sqrt{a}}{\sqrt{sm\beta}} = \frac{m\mu}{mr} \sqrt{a} = \frac{re}{cr} \sqrt{a} = \frac{re}{\frac{1}{2}d} \times \frac{a}{\sqrt{a}} = \frac{re}{d} \times \frac{a}{\frac{1}{2}\sqrt{a}} =) \frac{re}{d} \times t$, in a , Th. t , in all the pe 's (i. e. in the Cycloidal Arc πs) is = to all the re 's, or $\frac{1}{2}c \div d \times t$, in a , And t , in the Cycloid $\times d = t$, in the Axe $\times \frac{1}{2}c$.

P P

32. Hence

32. Hence, $c : d :: t$, of 1 Oscillation: t , of Descent in $\frac{1}{2}$ the Length (l) of the Pendulum. For $l =$ twice the Axe (by 72.4)

Put λ for the Length of a Pendulum, that measures $1''$ of Time in each Oscillation; Then $c : d :: v(t, 1'') : dv \div c =$ Time (τ) in $\frac{1}{2} \lambda$; And $(\tau^2 : v^2 ::) d^2 : c^2 :: \frac{1}{2} \lambda : b =$ Height fallen from in $1''$ of Time. But 'tis found, by means of Clocks adjusted by the Heavens, that $\lambda = 39,125$

Inches London Measure, Th. $b (= \frac{1}{2} \lambda \times c^2 \div 12 d^2) = 16,0895$ Feet, or 16 F. and about 1 Inch.

The Relistance of the Medium does somewhat vary the Time of Descent in a Cycloid; But in a Medium that does not Resist, the shorter Oscillations in a Cycloid are nearly *Isocronal*. And the greater the *Funipendulous* Body is, the less does the Medium Resist it: But then the greater is the Distance of the Centre of Gravity from the Centre of Oscillation (i. e. the Point whose Distance from the C. of Suspension is the Length of a Simple Pendulum, whose Vibrations shall be *Isocronal* to those of the given Magnitude, or that Point wherein all the Figure is supposed to be contracted, with the Forces, while it vibrates.) A Right Line parallel to the Horizon, about which the Oscillation is made, is called the *Axis of Oscillation*. And every right Line, or Plane passing thro' the Centre of Gravity is an *Axis*, or Plane of Equilibrium.

33. Given two Weights (W, w) and the Distance (Dd) between their Centres of Gravity D, d : Req'd. (g) their Common Centre of Gravity, or that Point in which all their Forces unite, or where, if they be joynly suspended, they'll produce the same Effect as they did separately. Let ($W : w :: dg :: Dg$, i. e.) $W + w : W :: Dd : Dg$, or $W + w : w :: Dd : dg$. Th. if there be any given Weights A, B, C , &c. and the Distance between their Centres of Gravity; Then (g) the Common Centre of Gravity of 'em all is easily found: For let a be the Centre com. to A and B , also b the Centre of $A + B$ and C , &c. And if any Weight A, B, C , &c. be apply'd to any Points of a Line suspended at a given Point, and distant from it by a, b, c , &c. Then will (d) the Distance of their Common Centre of Gravity (g) from the Point of Suspension be $=$ to $Aa + Bb + Cc$, &c. $\div A + B + C$, &c. or $M + m + m$, &c. $\div W + w + w$, &c. Also, if there be given several Weights A, B, C ,

A, B, C, &c. (in the same or different Planes) whose Distance from the Axis of Oscillation call a, b, c , &c. and the Distance of their Common Centre of Grav. from the same is d ; Then the Distance (δ) of the Centre of Oscillation from the Point of Suspension (or the Length of a Simple Pendulum that shall move as fast as a Pendulum composed of all the Weights) is equal to

$$\frac{Aa^2 + Bb^2 + Cc^2}{A + B + C}, \text{ \&c.} \div A + B + C, \text{ \&c.} \times d, \text{ that is,}$$

$$\frac{F + f + f, \text{ \&c.}}{M + m + m, \text{ \&c.}} \text{ Th. if the Weights are equal, and their Number } n, \text{ then } \delta = \frac{a^2 + b^2 + c^2}{n}, \text{ \&c.} \div nd.$$

34. In Quantities that are Suspended to, or do Oscillate about an Axis (A); Let d, δ be the Distance of the Centre of Gravity, or of Oscillation from (A) the Axis of Suspension, or of Oscillation: x the Abscissa of a Curve whose Ordinates (y) are parallel among themselves, and right to the Diameter; λ the Length of the Curve; a the Area adjacent to the Abscissa or its Parallel, and Flowing $\perp$ to A; s the Surface of a Solid generated by the Rotation of a Plane adjacent to the Abscissa or its Parallel, about any Side of the Curve's circumscr. Pgr. and Flowing $\perp$ to A; χ the Content of such a Solid; p the greatest Perp. from the Curve to A; let m represent either λ, a, s , or χ ; and ϕ the Fluent of any Fluxion: Then,

1. Where A is a Tangent to the Vertex of the Curve; $d =$

$$\phi, mx \div m; \delta = \phi, mx^2 \div \phi, mx.$$

2. Where A is the Base of the Curve; $d = pm - \phi, mx \div$

$$m; \delta = \phi, p - x \mid^2 m \div \phi, p - x m.$$

3. Where A is the Abscissa; $d = \phi, my \div m; \delta = \phi, my^2$

$$\div \phi, my.$$

4. Where A is parallel to the Abscissa; $d = pm - \phi, my$

$$\div m; \delta = \phi, p - y \mid^2 m \div \phi, p - ym. \text{ Examples in Case 1.}$$

If $y = x^n$, then $a = (yx) x^n$, th. $a = x^{n+1} \div n + 1$,

$$\text{Th. } d = \left(\frac{\phi, ax}{a} \right) \frac{n+1}{n+2} x; \text{ And } \delta = \left(\frac{\phi, ax^2}{\phi, ax} \right) \frac{n+2}{n+3} x.$$

P P 2

Alfo,

Also, bec. $\dot{\chi} = \dot{x}x^{2n} \times \overline{c \div r}$, & $\chi = cx^{2n+1} \div r \times 2n+1$

Th. $d = \left(\frac{\phi, \dot{\chi} x}{\chi} \right) \frac{2n+1}{2n+2} x$; And $\delta = \left(\frac{\phi, \dot{\chi} x^2}{\phi, \dot{\chi} x} \right) \frac{2n+2}{2n+1} x$

Note, A Surface, or Solid generated by the Uniform Rotation of a Line, or Surface about an Axe, is equal to a Surface, or Solid whose Base is that given Line, or Surface, and whose Altitude is = to the Periphery descr. by its Centre of Gravity.

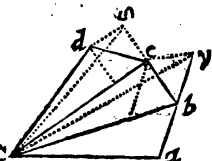
In a Cylinder (whose Height = b , Rad. of its Base = r) suspended by (P) the Extremity of the Axe, The Distance (gc) from g (the Centre of Grav.) to c (that of

Oscillation) is = $\frac{1}{3} Pg + r^2 \div 2b$. And in a very small Rod (PE) suspended by the End P , $gc = \frac{1}{3} gE$. But if PE be taken as the Diameter of a Sphere suspended by P , then $cg = \frac{2}{3} Pg$. And if a Sphere be suspended from an-

other Point (ω), then $cg = \frac{2}{3} Pg \times \overline{Pg \div \omega g}$, which gives the Centre of Oscillation in Order to adjust the Pendulum of a Clock. So that if ωg remain the same, then cg will be as Pg ; Th. if $\omega g = 39,125$ Inches, and $Pg = 1$, then $cg = \frac{1}{1.5}$ nearly; Also if $\omega g = 39,125$, and $Pg = 3$, then $cg = \frac{2}{1.5}$ nearly.

If (ωg) the Distance between the Centres of Suspension and Gravity be the same, then the Greater the Sphere is, the Slower will it Oscillate; But if neither the Bigness of the Sphere, nor ωg be alter'd, then, in Resisting, or Non-resisting Mediums, the Lighter the Sphere is, the slower, or quicker will it Oscillate. And without having Respect to the Centre of Oscillation, the Length of a Pendulum for determining any Time, or for any Parallel of Latitude, cannot be accurately found.

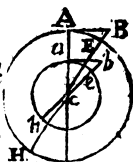
35. Suppose a Body in the 1st Moment of Time to run thro' ab , theref. the same Velocity continuing, it shall run in the 2d Moment thro' by ; but in b , sup. the Centripetal Force acting, draw $\gamma c \parallel bc$, and = to the Length thro' which the Body is carried towards C in a Moment, then bc is run thro' C in the 2d Moment; and the $\Delta Cb = \Delta \gamma Cb = \Delta bac$: Likewise in the next Moment, will the Body describe



ſcribe cd , making the $\Delta cCd = \Delta \gamma Cb$, and ſo on; Whence, equal Areas are deſcrib'd in equal Times. Let as well the Moments of Times, as the right Lines ab , bc , &c. be conceiv'd infinitely ſmall, then the Polygon $abcd$ will become a Curve Figure, from whole Tangent the Body iſperpetually retracted by the *Vis-Centripeta*; And the Areas deſcrib'd by the Radii drawn from the Centre (C) to the Body ſhall be Proportional to the Times of Deſcription.

And if a Body move in a Curve Line, and by Rays drawn to the ſame Point deſcr. Areas Proportional to the Times, it is retracted from the Tangent to this Curve by a Centripetal Force tending to this Point.

36. Let C be the Centrifugal or Centripetal Force, T the Periodical Time, V the Velocity; Sup. AB a Tangent, BH a Secant, then $BE = AB^2 \div BH$ (by 64.19) But if AB be infinitely ſmall, then will $BE = AE^2 \div EH$; Th. (if AE, ae be Arcs deſcrib'd by two movable Bodies in the ſame H. Time; Then $C : c :: (AB^2 \div 2R : ae^2 \div 2r ::) V^2 r : v^2 R :: R^2 : r^2$; Th. if $T^2 : t^2 :: R^2 : r^2$, then $CR = cr$; And $V = v$. If $T^2 : t^2 :: R^3 : r^3$, then $CR^3 = cr^3$; And $V \times \sqrt{R} = v \times \sqrt{r}$.



'Tis obſerv'd, That the Primary Planets do deſcribe equal Areas about the Sun in equal Times; And that the Squares of their Periodic Times are as the Cubes of their Diſtances from it; Th. 'tis the Sun that is the Centre of the Planetary System.

It follows alſo, that the Force of Gravity of the Heavenly Bodies is Reciprocally as the Squares of their Diſtances from the Centre. And $V : v :: r : \sqrt{R} :: \sqrt{R} : R$, th. the nearer the Centre, the greater the Velocity.

If 2 Bodies deſcribe equal Areas (CAE, cae) in equal Times, or if $CA \times AE = ca \times ac$, i. e. $VR = vr$, and $V^2 R^2 = v^2 r^2$ Then $C : c :: (V^2 \div R : v^2 \div r ::) r^2 \div R : R^2 \div r ::) r^3 : R^3$.

37. The Ls, A, a (which, a Body deſcr. about the Centre of Attraction) are as r^2 to R^2 . For $V : v :: RA : ra$, th. $A : a :: (Vr : vR) :: (bec. if T = t, then V : v :: r : R) r^2 : R^2$.

38. If $v, V =$ Velocity acquired in the Time t, T , $s =$ Space run, ſuppoſing, $t^3 : T^3 :: v^3 : V^3$; $S =$ Space run by an Equable Motion in the Time T, with Velocity V, (let

m

$m \div n = x$) Then $t = Tv^x \div v^x$, th. $\dot{t} = x\dot{v} Tv^{x-1}$
 $\div V^x$, but $\dot{s} = v\dot{t} = x\dot{v} Tv^x \div V^x$, th. $s = x \div x + 1$

$xTv^{x+1} \div V^x =$ (putting V for v) $m \div m + x \times TV$;
 And $S = VT$, Th. $s : S :: m : m + x$; Or if $T : t :: V : v$, then $s : S :: 1 : 2$.

Th. if S , and Σ , = *Space* run by an *Equable*, and an *Equably Accelerated Motion* in the *Time* t , and vt , Then
 $t^2 : v^2 t^2 :: \frac{1}{2}S : \frac{1}{2}Sv^2$, and $S : (\frac{1}{2}Sv^2) \Sigma :: 2 : v^2$.

39. Let the Arc a be descr. by a Body revolving in a Circle with an *uniform Motion* in the *Time* t ; and the Rt. Line b descr. by its Descent with *Accelerated Motion* in the *Time* vt ; then (bec. $a : b :: 2 : v^2$), $a = 2b \div v^2$; but the *Centrifugal Force* (c) in the time t is $a^2 \div 2r$ (by 36.) $= 4b^2 \div 2rv^4$, Therefore the *Space* run by the *Centrifugal Force* (c) in the *Time* t : sp. run by (*Gravity*) g in *Time* vt ($: 4b^2 \div 2rv^4 : b$) $:: 2b : rv^4$.

Th. if a Body (whose *Centrifugal Force* is c , *Gravitating Force* g) moves uniformly in a *Periphery* (p) whose *Rad.* is r , with a *Velocity* equal to that acquired by falling the *Height* b ; Then shall $c : g :: 2b : r$, Th. if $b = \frac{1}{2}r, g = c$.

40. If any two *Pendulums*, carried with a *Conic Motion*, descr. *Peripheries* (P, p) whose *Radii* are R, r ; The *Times* (T, t) of *Description* shall be as the *Square Roots* of the *Altitudes* (A, a) of the *Cones*. For $g : c :: a : r$, th. $c = (rg \div a) 2bg \div r$, and $b = (r^2 \div 2a) v^2$, th. $v = r \div \sqrt{2a}$, but

$(p \div v) p \times \sqrt{2a} \div r =$ *Time* t in p ; And $(p \times \sqrt{2a} \div r :$

$p \times \sqrt{2A} \div R ::) \sqrt{a} : \sqrt{A} :: t : T$. Th. if $A = a$, then $T = t$.

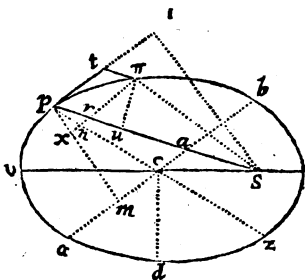
41. In a *Pendulum* carried by a *Conic Motion*; *Time* in descr. the least *Periphery* : *Time* in falling an *Alt.* = twice the *Length* (L) : *Circumf.* : *Diameter*. For v , in $2A$ is $\sqrt{2A}$, and *Time* is $4A \div \sqrt{2A}$ or $2\sqrt{2A}$, And $2\sqrt{2A} :$

$p \times \sqrt{2A} \div R :: 2R : P :: Diam. : Circumf.$ But (bec. R is sup. infinitely small) $L = A$, Th. Sc. Th. the *Time* in describing the smallest *Circuit* is = to the *Time* of the two smallest *Oscillations* of the *Pendulum*.

42. If

43. If a Pendulum descends thro' a Quadrantal Arc, the *Vel.* in the lowest Point is = to that acquired in falling thro' r , th. $c = (2hg \div r) = 2g$, Th. the Force in the lowest Point is $= (2g + g) 3g$.

45. If a Body in revolving about a Centre (s) describes a Curve Line $vp\pi$; Let pt be a Tangent, $t\pi \parallel$ and $\pi u \perp ps$; Now time (T) is as $\Delta s\pi p$ (by 35) or as $\pi u \times sp$, and if T be given, the Lineola $t\pi$ is as C (the Centripetal Force), but if C be given, $t\pi$ is as T^2 , or as $\pi u^2 \times sp^2$, therefore universally, $t\pi$ is as $C \times \pi u^2 \times sp^2$, th-



Example. *If a Body revolve in an Ellipse; to find the Law of the Centripetal Force tending to the Focus.* Draw the Diameter $p\zeta$, and ab Conjugate to it, cutting ps in a ; Let $pm \perp ab$, and $\pi x \parallel pt$ cutting ps, pc, pm in r, n, x : Then $ap = cv$ (by 68. 4.) and $cb \times pm = cd \times cv$ (by 68. 28.) also (bec. $\pi t : tp :: 1 : \infty$) $\pi r = \pi x$, and $zn = zp$; But $\pi u^2 : \pi r^2 :: pm^2 : pa^2 :: pm^2 : cv^2 :: cd^2 : cb^2$, and $\pi r^2 : zn^2 :: (\pi n^2 : zn^2 ::) c\dot{o}^2 : pc^2$, th. $\pi u^2 : cd^2 :: zn^2 : cp^2 :: np : (cp^2 \div zp)^{\frac{1}{2}} cp :: pr : (\frac{1}{2}pa) :: t\pi : \frac{1}{2}cv$, th. $t\pi : \pi u^2 = \frac{1}{2}cv \div cd^2$, th. C or $t\pi \div \pi u^2 \times sp^2$

And by a like reasoning, 'tis readily found, that if a Body move in an *Hyperbola*, or *Parabola*, the *Centripetal Force*

Force tending to the Focus, shall be reciprocally as the Square of the Distance. But if the Centripetal Force should tend to the Centre of the Ellipse or Hyperbola (which is no where found) it would be directly as the Distance.

Hence, if a Body move in a Rt. Line from any place, with any Velocity, and with a Centripetal Force that is reciprocally as the Square of the Distance, it shall move in some of the Conic Sections, whose Focus is in the Centre of Forces.

46. Let A, a be the Areas describ'd in different Orbs or Ellipses E, e , whose Transv. Axes are D, d , Parameters P, p , let T, t be the Periodical Times. Then if several Bodies revolve about a Common Centre, and that C is as $1 \div ps^2$, Then (bec. $P = (cd^2 \div \frac{1}{2}cv, \text{ by } 68.5) \pi u^2 \div \pi r$ and in a given Time, πr is as C (by 45) or as $1 \div sp^2$, th. P is as $(\pi u^2 \times sp^2) = A^2$, That is, $P : p :: A^2 : a^2$, Or the Parameters of the Orbs shall be as the Squares of the Areas describ'd in the same time.

47. Th. if $T = t$, A is as $\sqrt{P}$, but if $P = p$, A is as T , Th. universally, $A : a :: \sqrt{P} \times T : \sqrt{p} \times t$, but $A : a :: \sqrt{P} \times \sqrt{D^3} : \sqrt{p} \times \sqrt{d^3}$ (by 68.26) th. $T^2 : t^2 :: D^3 : d^3$. That is, the Squares of the Periodic Times are as the Cubes of the Transverse Axes, or mean Distances from the Centre of Attraction.

48. The same things being supposed (as in 45,) draw $st (= \delta) \perp pt$ (the Tangent,) Then the Velocity (V) of the Body is as the little Arc $p\pi$ descr. in a given Time, i. e. as pt or πr , but $\pi r : \pi u :: sp : st$, th. pt is as $\pi u \times sp \div st$, i. e. V is as $(A \div \delta) \sqrt{P} \div \delta$ or $\sqrt{P \div \delta^3}$.

83. Of the Motion of Projects.

1. A Body being Projected by any Force moves in the Curve of a Parabola, unless so far as the Resistance of the Medium binds it. Thus, if HK be $\perp$ to the Horizon, and any Rt. Lines $\mu v, mV$ be such, as that in the Time the Body by its Projectile Motion arrive to μ, m , it may fall, by its Descending Motion, the Length $\mu v, mV$; Then (by 82.1.) $\mu v : mV :: H\mu^2 : Hm^2$, or $Hk : HK :: kv^2 : KV^2$, which is the Property of the Parabola.

2. The Horizontal Distances (H, h) of Projects, made with the same Velocities, at several Elevations (E, e) are as

as the Sines of the double *L*s of Elevations. For (if $s = \text{Sine}$, $\zeta = \text{Co-Sine}$) $x(\text{RL}) = bs \div \zeta$, and $y(\text{HL}) = br \div \zeta$

$$\text{th. } (y^2) \frac{pbs}{\zeta} = \frac{b^2 r^2}{\zeta^2}, \text{ th. } b = \left(\frac{ps\zeta}{r^2} = \frac{\frac{1}{2}p}{r} \times \frac{2s}{r} = (\text{bec.} \right.$$

$$2s\zeta \div r = s, 2L, \text{ by } 15.65.) s, 2L \times \frac{1}{2}p \div r, \text{ Th. } H : b :: s, 2LE : s, 2Le.$$

2. Hence, when $Le = 45^\circ$, the Sine of its double is the greatest Sine; but the Ranges are as the Sines of the double *L*s, *Th. the greatest Random is at 45° Elevation.* And bec. at 45° Elevat. $s, 2L = r$, th. $b(\text{or } g) = \frac{1}{2}p$, i. e. *The greatest Random is equal to $\frac{1}{2}$ the Parameter.* Also, the Ranges equally distant above and below 45° are equal.

3. The Altitudes (*A, a*) of Projections, made with the same Velocity, at several Elevations (*E, e*) are as the Versed Sines of the double *L*s of Elevations. For $s : s :: (b)$

$$\frac{ps\zeta}{r^2} : (x) \frac{ps^2}{r}, \text{ And } a(\text{MV} = \text{Vm}) = \frac{1}{2}x = \frac{ps^2}{4r^2} = \frac{2s^2}{r} \times$$

$$\frac{\frac{1}{2}p}{4r} = (\text{bec. } \frac{2s^2}{r} = v, 2L, \text{ by } 15.65.) v, 2L \times \frac{\frac{1}{2}p}{4r} (= v, 2L \times \frac{p}{8r})$$

Th. $A : a :: v, 2LE : v, 2Le$. And since $2r = \text{greatest vers'd Sine}$, th. $r : 2r :: \frac{1}{2}p : \frac{1}{4}p = \text{greatest Altitude} = \frac{1}{2}$ the greatest Random.

4. The Times (*T, t*) of the Flights of a Project thrown with the same Velocity at different Elevations (*E, e*) are as the Sines of the Elevations. For let $y(\text{HL})$ represent the Time, then $r : r :: (b) ps\zeta \div r^2 : ps \div r = y$, th. $r : p :: s : y :: S : Y$, i. e. $T : t :: s, LE : s, Le$.

5. Given (*b*) the Horizontal Distance of an Object, (*e*) the *L* of Elevation, (*n*) the Ascent or Descent. Req'd. the greatest Random ($g = \frac{1}{2}p$), and Velocity.

$$1. \text{cs, } e : b :: s, e : x \pm n, \text{ and } b^2 + \overline{x \pm n}^2 = y^2. \text{ Or } r : s, e :: b : y, \text{ and } y^2 : x = p = 2g.$$

$$2. \text{ Since } x = \text{Fall in the Time } y, \text{ th. } 193 \text{ Inches} : x :: 1'' : \frac{1}{193} x \times 1'' = \text{Sq. of the Time in } x, \text{ Th. Velocity } (S \div T) = y \div \sqrt{\frac{1}{193} x} = \sqrt{px} \div \sqrt{\frac{1}{193} x} = \sqrt{193p}.$$

6. Since the Force or Velocity of a Project, describing a Curve Line, is compounded of the Uniform Velocity in y , and of that Uniformly Accelerated in x ; which will carry it but $\frac{1}{2}$ the Space that it would run uniform-

ly with the last acquir'd Velocity in the same Time, (by 33. 82.) Therefore, the Velocity in any Point (P) of the Curve is to the Velocity impress'd in H , as the *Tangent* to the *Ordinate*; or as the Lengths of *Tangents*, to the Points P , and H , intercepted between the *Diameters* to those Points; or as the *Secants* of the Ls made by those *Tangents* produc'd, and the Horizontal Line. Th. the *Least Force* is at the Vertex of the Curve, or at the Height of $\frac{1}{4}p$. And at equal Distances from the Vertex, the Forces are equal. Also, *Least Force* : *Impress'd Force* at H :: r : f , *L.e.*

7. Given, L, e, g ($\frac{1}{2}p$), n , and L, a ; Required b .

Bec. $ys \div r = x + n$, th. $x = ys + rn \div r = y^2 \div p$, Th. $y^2 - spy = +prn$; And $r : y :: cs, e : h$.

8. Given, b, n, g ($\frac{1}{2}p$); Required E, e , (whose *Tangent* call t .) Since, $bt \div r. (b) = x + n$, th. $b + n = x$; but $b + n \times p (=xp = y^2) = b^2 + b^2$, and $b^2 = bp + np - b^2$. Therefore, $t \div r = p \div 2b + (f)$
 $\frac{p^2 + 4pn \div 4b^2 - 1}{2}$ or $t = r \times \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + pn - b^2} \div b$.

Th. $b^2 : \frac{1}{2}p :: \frac{1}{2}p + 2n : p^2 + 4pn \div 4b^2 =$ to a f^2 , whose *Tangent* is f , And $f \pm \frac{1}{2}p \div b = t, E, e$, sought.

1. Th. if $f > \frac{1}{2}p \div b$, or if $pn > b^2$, then, in *Descents*, the Direction of the lower Elevation will be below the *Horizon*. But if $pn = b^2$, the Direction will be *Horizontal*, and $pr \div b =$ *Tangent* of the upper Elevation. And if $b^2 \pm pn > \frac{1}{4}p^2$, the Object (in *Ascents*, or *Descents*) is beyond the reach of the Project thrown with that Velocity.

2. If $\frac{1}{4}p^2 = b^2 + pn$, in *Ascents*, or *Descents*, there can be but one Elevation (whose *Tangent* is $pr \div 2b$) that will reach the Object. Th. $n = \frac{1}{4}p - b^2 \div p$, or $b^2 \div p - \frac{1}{4}p$, which determines the utmost Height of any jet upon each given *Horizontal* Distance. And $\frac{1}{2}p = \sqrt{b^2 + n^2} + n =$ *Horiz. Range* at 45° of a Project, thrown with the least Velocity capable to reach the

$\pm 2nx = px \mp 2nx \pm 2nx =) px$; i. e. $Hl^2 = p \times Pl$.
th. the Curve will pass thro' P. Or bec. $u = (w^2 - r^2)^{\frac{1}{2}}$
 $= \frac{1}{2} p \mp n \sqrt{b^2 + n^2}^{\frac{1}{2}} =) \frac{1}{4} p^2 \pm pn - b^2 \sqrt{\frac{1}{2}}$ there-
fore $Ol = \frac{1}{2} p \pm u$, and $b:r :: \frac{1}{2} p \pm u$: (t, E, c)

$r \times \frac{1}{2} p \pm \frac{1}{4} p^2 \pm pn - b^2 \sqrt{\frac{1}{2}} = b$. Otherwise, Let CH
1 HP, AH (1HO) = $\frac{1}{2} p$, and AC || HO; From C, with
CH, describe an Arc cutting (if possible) OB in l, l; Then
will Hl be the Lines of Direction. For since AC =
 $\frac{1}{2} pn \div b (= c)$, theref. $CB^2 = c^2 \mp pn + b^2$, and Cl^2
(CH²) = $c^2 + \frac{1}{4} p^2$, th. $Bl = \sqrt{\frac{1}{4} p^2 \pm pn - b^2}^{\frac{1}{2}}$ And

$b:r :: \frac{1}{2} p \pm u$: (t, E, c) $r \times \frac{1}{2} p \pm u \div b$. Hence the 5th,
7th, and 8th Proposition may be Arithmetically Solv'd thus,

$$1. (\frac{1}{2} E + c = 90^\circ - \frac{1}{2} \alpha, \text{ th.}) 90^\circ - \frac{1}{2} \alpha \sin c = \frac{1}{2} E - c$$

and $v, \alpha (AZ) - v, E - c (BZ) = q (AB)$, Then $q : s, \alpha (AH)$,

$$:: b : \frac{1}{2} p = g.$$

2. Having found q , then $s, \alpha : q :: \frac{1}{2} p : b$.

3. Let $\frac{1}{2} p : b :: s, \alpha (AH) : q (AB)$, and $v, \alpha - q$
(AZ - AB) = $v, E - c (BZ)$, but $\frac{1}{2} E + c (=$
 $\frac{1}{2} 180^\circ - \alpha) = 90^\circ - \frac{1}{2} \alpha$; And $\frac{1}{2} E + c \pm \frac{1}{2} E - c$
 $= E, c$, required.

And because of the Air's Resistance, the Line of Pro-
jects is not exactly Parabolical, but rather a kind of an
Hyperbola; which, if consider'd, and apply'd to Practice,
would render the Computation far more operose, and
the very small Difference (as Experience shews in Heavy
Shot) would, in a great Measure, lessen the Elegancy of
the Demonstrations given by Accounting for it; since
the former Rules are sufficiently exact and easie for
Practice.

The Theory of the Motion of Projects is so perplex'd a
Subject, and depends so much upon Physical Observations,
that such Accuracy cannot be expected therein, as don't
require some Allowances. But the Resistance of the Me-
dium, and other Accidental Impediments, may be in some
measure

measure Rectified, by ſuppoſing the Shot to move in a Right Line to a certain Diſtance (d) from the *Axis* of the Gun, and afterwards to deſcribe the Curve of a *Parabola*.

Given, E, e ; H, h : Req'd. p, d . Since $d+y$, and x are given (by *Trigonom.*) alſo $y = \sqrt{px} \sqrt{x}$ (by 36. 68), and $d = D$; th. $D + r \sin d + y = r \sin y = \sqrt{X \sin \sqrt{x}} \times \sqrt{p}$, Th. $p = \frac{D + r \sin d + y}{\sqrt{X \sin \sqrt{x}}}^2$ And $d = d + y - \sqrt{px} (y)$

Given $d, \frac{1}{2}p, Le, n$; Req'd. b . Since $r : d + y :: s : ds + ys \div r (= x \pm n) = y^2 \div p \pm n$, th. $ry^2 - psy = pds \mp rnp$; (let $r : s :: p : q$, or $rq = sp$) and $y^2 - qy = dq \mp np$, th. y is given; And $r : d + y :: cs, e : b$ ſought.

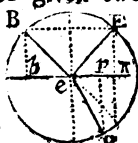
Given $d, b, \frac{1}{2}p, n$; Req'd. Le . Bec. $x \pm n = (y^2 \div p \pm n) \frac{y^2 \pm np}{p}$, th. $y^2 \pm np \div p \pm b^2 = y + d^2$, th. $y^4 \pm 2np - p^2 \times y^2 - 2dp^2y = d^2 - b^2 - n^2 \times p^2$, th. y is given; And $d + y : r :: b : cs, Le$ ſought.

84. Of Optics.

That Science, which accounts for the various Appearances of Objects, from the Reflexions, Refractions, and Inflexions of the Rays of *Light*, is called *Optics*.

Experience ſhews that the Light of the *Sun* conſiſts of Rays that are differently *Reflexible* and *Refrangible*. The Angle comprehended between the Incident, Reflected, or Refracted Ray, and the Perpendicular to the Reflecting, or Refracting Surface is called the *Angle of Incidence*, *Reflexion*, or *Refraction*. And theſe three Angles lie in ~~one~~ and the ſame Plane. The *Refraction* out of a Rarer Medium into a Denſer is made towards the Perpendicular; and the contrary.

And, ſuppoſing Nature's Method of working to be the moſt eaſie and expedite, if there be given two Points (B, P) in Mediums of different gi-
ven Denſities, and the Poſition of the Plane ($b\pi$) dividing thoſe Mediums; Req'd. the Point (e) in that Plane, thro' which the Body paſſes by taking the ſhorteſt Time to move from B to P .



Let

Let $Be, eP = s, S$; $Bb, Pp = c, C$; $bp = a, be = r$, then $pe = a - r$; let $v, V =$ Velocity in s, S : Th. since r , in s :

$r, S :: sV : Sv$, th. r , in $s + S$ is $\sqrt{c^2 + r^2}^{\frac{1}{2}} \times V +$

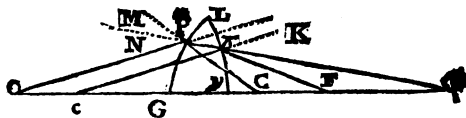
$\sqrt{c^2 + a - r^2}^{\frac{1}{2}} \times v$, which if a Minimum, then $V\ddot{r} \div s$

$+ v\ddot{r} - v\ddot{a} \div S = 0$. Th. $SV\ddot{r} = sv \times a - r$, th. $SV : sv :: a - r : r$, Th. (supposing $s = S$) $v : V :: r : a - r$ ($:: m : n$) i.e. The Sines of the $\angle$ of Incidence, and of the Refracted $\angle$ are directly as the Velocities, or reciprocally as the Densities of the Mediums.

But if a Body moving from B to e be Reflected to E , bec. it continues in the same Medium, th. $v = V$, conseq. $s : S :: r : a - r$, and Ls at b and π are right, th. $\angle Beb = \angle Ee\pi$, i.e. The $\angle$ of Incidence is $= \angle$ of Reflexion.

Homogeneous Rays, (or those of like Refrangibility) which flow from several Points of any Object, and fall almost Perpendicularly on any Reflecting, or Refracting Surface, shall afterwards Diverge, or Converge, to, or from so many other Points, or be Parallel to so many other Lines, either accurately or very near. The same happens if the Rays be reflected or refracted successively by two or more Surfaces. That Point to, or from which Rays Converge, or Diverge is called the Focus.

Given a double Convex Lens ($GL\gamma$) whose Thickness is $G\gamma (=r)$, and O a Point or Object in the Axis of the Lens; the Ratio of Refraction being as m to n : Reqd. the Point (F) at which the Beams that are nearest the Axis of the Lens are collected.



Let $OG (=OP) = d$, the Distance of the Object, $CG (=CP) = r$, and $c\gamma (=c\pi) = e$, the Radius of the Segment towards and from the Object: And let S, L Incid. (OPM): S , Refr. L (NPM or $CP\phi$) :: $m : n$; then, in very small Ls , L Incid.: Refr. $L :: m : n$. Theref. $d : r :: LC : LO$; th. $d + r$ is as the L Incid. (OPM) and $m : n :: d + r : n d + nr \div m$, which is as the Refr. L ($CP\phi$)

$\angle (CP\phi)$; But $\angle PCO = CP\phi$ (or $d - nd + nr \div m$)
 $= \angle PO\phi$. And the $\angle \phi : \angle O :: PO : P\phi (= G\phi)$

$= mrd \div md - nd - nr$, (that is, the Beams from O
 by the first Refraction will be collected in ϕ .) If nr be
 greater, or equal to $md - nd$, then the Beams after Re-
 fraction go Diverging from, or Parallel to the Axis,
 and the Point ϕ is on the same side beyond P , or Infinitely
 distant. Also (if $\delta = G\phi - G\gamma = \phi\gamma$ or $\gamma\phi$) $\delta : \phi$
 $:: \angle c : \angle \phi$; th. $\delta + \phi$ is as the $\angle$ Incid. ($\angle K\pi\phi$), and

$n : m :: \delta + \phi : m\phi + m\delta :: n$, which is as the Refr. $\angle (K\pi F)$

But $\angle K\pi F = \angle KcF$ (or $m\phi + m\delta \div n - \delta$) $= \angle cF\pi$.

And the $\angle cF\pi : \angle KcF :: \pi c : \pi F = n\phi \div$

$m\delta - nd + m\phi =$ (if $a = n \div m - n$) unto

$$\frac{madr - d\phi - ar\phi \times m}{m \times dr + d\phi - ar\phi - md - nd - nr \times t} = f = \text{(if } t \text{ be rejected)}$$

$adr\phi \div dr + d\phi - ar\phi$. Whence, any four of these five
 Quantities (d, r, ϕ, a, f) being given, the fifth is readily
 found. And,

In Diverging Rays, falling on a double Convex, or
 Concave (where it is $+d$, & $\pm r, \pm \phi$) $\pm adr\phi \div$

$dr + d\phi \mp ar\phi = \pm f$; But, in the Convex, if $ar\phi$
 $> dr + d\phi$, then 'tis $-f$.

In Converging Rays, falling on a double Convex, or
 Concave, (where 'tis $-d$, and $\pm r, \pm \phi$) $- adr\phi \div$

$+ dr + d\phi - ar\phi = \pm f$; But in the Concave, if $ar\phi$
 $> dr + d\phi$, then 'tis $+f$.

In Diverging, or Converging Rays, falling on a Menif-
 cus, $\mp adr\phi \div \mp dr + d\phi + ar\phi = \pm f$, whilst $dr -$

$d\phi >$ or $< ar\phi$; But if $dr - d\phi <$, or $> ar\phi$, 'tis $-f$.

In Parallel Rays, falling on a Double Convex, (where
 $d = \infty$) $f = a\phi \div r + \phi$ or (in Glass, where $m : n :: 3$

$: 2$) $= 2r\phi \div r + \phi$, Th. if $r = \phi$, then $f = r$.

In Diverging Rays, on a Plano-Convex, (where $r =$
 ∞) $f = ad\phi \div d - a\phi$.

If, $t = 2r$, & $r = \phi$, and $d = \infty$, then $f =$

$2nr - nr \div 2m - 2n$, = (in a Sphere of Glass)

$\frac{1}{2} r$, or in a Sphere of Water, (where $m : n :: 4 : 3$) $f = r$.

And

And wherever the Rays, which come from all the Points of any Object, meet again in so many Points, after they have been made to Converge by Reflexion or Refraction, there will they make a Picture of the Object. Now, a Lens being given, to find the Distance, whereas an Object being plac'd, shall be Represented as large as the Ob-

ject it self; Here $d = f$, Th. $2 ar \div r + e = d$; or if $r = e$, then $d = ar$; or, in Glasses $d = 2r$: In a Plano Convex, $d = 2ar$, i. e. in Glasses, $d = 4r$, This is of singular use for Drawing things in their just Magnitude, by Transmitting the Species by a Glass into a dark Room, whereby, not only the True Figure and Shades are perfectly given, but also the Colors nearly as Vivid as the Life. 'Tis also easie from hence, to Magnify or Diminish an Object in any given Proportion. An Object seen by Reflexion or Refraction, appears in that Place from whence the Rays, after their last Reflexion or Refraction, diverge in falling on the Spectator's Eye. Whence, As the distance of the Object from the Lens, is to the Diameter of the Object's Magnitude, so is the Distance of the Image to its Diameter. And if the Object be seen, thro' two or more Glasses, every Glass shall make a New Image, and the Object shall appear in the Place, and of the Bigness of the last Image: Upon this depends the Theory of Microscopes and Telescopes.

Having given (r) the Rad. of one Segment of a Lens; to find the Rad. of the Convexity or Concavity, necessary to make a vastly distant Object be represented at a given Focus. Since r , a , f , and d , are given, Th. $drf \div adr - df + arf =$ (because $d = \infty$) $rf - ar - f = +e$, or (if $f > ar$) $= -e$, and $rf \div f - ar =$ Radius of the Concavity.

NOTE, That by putting $2rx - xx =$ to the Value of y^2 in any Curve, and rejecting the Terms concern'd with the Powers of x , the Radius of a Circle Equicurve with that of the Lens, at the Vertex, is determin'd, (and in the Conic Sections will be $= \frac{1}{2}$ the Parameter); this substituted in the room of r , e , will give the Theorem for a Lens of any given Figure. And where the Rays are Reflected from (GL) the Concave Surface of a Speculum, (bec. 'tis $+d$, $+m$, $-f$, $-r$, $-n$, and $m = n$,) $f = dr \div 2d - r$.

FINIS.



